Editorial Board

Editor in Chief
Prof Kenneth K. Nwabueze; UPNG, Port Moresby, Papua New Guinea;
E-mail: knwabueze@gmail.com
(Finite groups and representations)

Advisory Editors
Prof John Pumwa; PNG Uni of Technology, Lae, PAPUA NEW GUINEA;
E-mail: pumwa@mech.unitech.ac.pg
(Engineering Mathematics)

Prof. Fred Van Oystaeyen; University of Antwerp, Belgium;
E-mail: fred.vanoystaeyen@ua.ac.be
(Non-commutative algebraic geometry)

Prof. M. A. Satter; PNG Uni of Technology, Lae, PAPUA NEW GUINEA;
E-mail: masatter@mech.unitech.ac.pg
(Engineering Mathematics)

Prof. Michel Waldschmidt; Université Pierre et Marie Curie, FRANCE.
E-mail: michel.waldschmidt@upmc.fr
(Number Theory)

Prof. Claude Levesque; University of Laval, Quebec, CANADA.
E-mail: Claude.Levesque@mat.ulaval.ca
(Algebraic Number Theory)

Professor Aderemi Kuku; Grambling State University, USA.;
E-mail: kukua@gram.edu
(Algebraic K-theory)

Editors
Professor S. Caenepeel, Vrije Universiteit Brussel, Belgium
E-mail: scaenepe@vub.ac.be
(Algebra)

Professor Daniel Makinde; Cape Peninsula University of Technology, S. Africa;
E-mail: makinded@cput.ac.za
(Fluid Mechanics)

Professor Omar Kihel; Brock University, Canada;
E-mail: okihel@brocku.ca
(Cryptography, and Number theory)

Professor Walter Roth; Uni of Brunei, Brunei;
E-mail: walter.roth47@gmail.com
(Functional Analysis)
Assoc. Prof. A. Moshi; PNG Uni of Tech, Lae, Papua New Guinea;
E-mail: hmoshi@cms.unitech.ac.pg
(Graph Theory)

Assoc. Professor MGM Khan; University of the South Pacific, Fiji;
E-mail: mgm.khan@usp.ac.fj
(Operations Research; Mathematical Programming, Bio-statistics)

Professor Viorel Barbu; Al. I. Cuza" University, Iasi, Romania.;
E-mail: vb41@uaic.ro
(Control Theory and Stochastic PDE)

Professor Kay Owens; Charles Sturt University, Australia,
E-mail: kowens@csu.edu.au
(Mathematics Education)

Prof. Cecilia Nembou; Divine Word University, PNG;
E-mail: cnembou@gmail.com
(Applied Mathematics)

Prof. S. M. Uppal; Jomo Kenyata Uni. of Agric. & Tech. Kenya;
E-mail: smuppal@gmail.com
(Mathematics Education)

Assoc. Prof. Subhash C. Dey; PNG Uni. of Tech, Lae, Papua New Guinea.
E-mail: sdey@ap.unitech.ac.pg
(Mathematical Physics)

Prof. Mihail Ursul; PNG University of Technology, Lae, Papua New Guinea.
E-mail: mursul@cms.unitech.ac.pg
(Topological Groups)

Prof Yinhuo zhang; University of Hasselt, Belgium;
E-mail: Yinhuo.Zhang@uhasselt.be
(Hopf Algebras)

Assoc. Prof. Ora Renagi; PNG University of Technology, Lae, Papua New Guinea.;
E-mail: orenagi@ap.unitech.ac.pg
(Mathematical Physics)

Assoc Prof. Samuel Kopamu, University of Goroka; Papua New Guinea;
E-mail: samkopamu@gmail.com
(Algebraic Semi groups)

Assoc. Prof. Praveen Pandey; PNG University of Technology, Lae, Papua New Guinea.
E-mail: ppandey@mech.unitech.ac.pg
(Engineering Mathematics)

Professor Stanford Shateyi; University of Venda, South Africa;
E-mail: Stanford.Shateyi@univen.ac.za
(Applied Mathematic)

**Scope and aims of the South Pacific Journal of Pure and Applied Mathematics**

The South Pacific Journal of Pure and Applied Mathematics published biannually publishes very high original research papers, survey papers, expository papers, research announcements describing new results, short notes on unsolved problems in all areas of pure and applied mathematics. Selection of papers for publication is on the basis of reports from referees commissioned by the editorial board. Manuscript submitted to this journal will be considered for publication with the understanding that the same work has not been published and not under consideration for publication elsewhere.

**Information for Authors**

**Preparation of Papers:** All papers must be in English, French, or German. Preparation of the manuscript in some form of TeX, such as LaTeX, AmS-TeX, or AmS-LaTeX, is highly recommended. Each manuscript is required to contain an abstract, clearly separated from the rest of the paper, which will be printed immediately after the title. On the title page, the following information should be given: AMS subject classifications number and key words and phrases.

**Submission of Papers:** A PDF file of the manuscript can be submitted by e-mail to the editorial office knwabueze@gmail.com or to any member of the editorial board. If accepted for publication, the source file of the manuscript should also be sent to the editorial office.

**Contacting and Subscribing**

All communications with this publication should be addressed to the Editor in Chief or any member of the editorial board.
On the number of solutions of the Diophantine equation \( y^2 = n(x(Ax^2 \pm C)) \)

Wenquan Wu
Department of Mathematics and Finance, ABa Teacher's College, Wenchuan, Sichuan, 623000, P. R. China
E-mail: wwq681118@163.com

Alain Togbé
Mathematics Department, Purdue University North Central
1401 S, U.S. 421, Westville IN 46391 USA
E-mail: atogbe@pnc.edu

Bo He
Department of Mathematics and Finance, ABa Teacher's College, Wenchuan, Sichuan, 623000, P. R. China
E-mail: bhe@live.cn

Shichun Yang
Department of Mathematics and Finance, ABa Teacher’s College, Wenchuan, Sichuan, 623000, P. R. China.
E-mail: ysc1020@sina.com

Abstract
In this paper, using deep results of Ljunggren, Cohn, Luca and Walsh, Yuan, Luo and Yuan et al on binary quartic equations \( Ax^2 - By^4 = \pm 1, \pm 4 \), we get a sharp upper bound of the number of solutions of the Diophantine equation \( y^2 = n(x(Ax^2 \pm C)) \), where \( C = \pm 1, \pm 2, \pm 4 \).

1 Introduction and main result
Let \( \mathbb{Z}, \mathbb{N}, \mathbb{P} \) be the sets of all integers, positive integers, primes respectively, and let \( a, b, c, d \) be integers. The elliptic equation
\[
y^2 = ax^3 + bx^2 + cx + d, \quad x, y \in \mathbb{N}
\]
is an important class of Diophantine equations. The literature is very rich, see for examples [2], [21], [22]. In 1923, Mordell [21] discussed this equation and gave upper bounds of the number of solutions. Subsequently, Siegel [22]
used a method of Diophantine approximation to give an upper bound of the number of solutions. But his method is non-effective. In 1968, Baker [2] used a lower bound of linear forms in logarithms of algebraic numbers to give an upper bound of the solution number of this equation that can be effectively calculated. Recently, Draziotis [12], Draziotis and Poulakis [13], [14], Feng [15] considered a series of special cases of equation 1.

In 1985, Cassels [5] proved that the equation

\[ y^2 = 3x(x^2 + 2) \]  

(2)

has only positive integer solutions \( x = 1, 2, 24 \). In 2005, Luca and Walsh [18] used classical results of Ljunggren and proved that the equation

\[ y^2 = nx(x^2 + 2) \]  

(3)

has at most \( 3 \cdot 2^{\omega(n) - 1} \) positive integer solutions \( (x, y) \), where \( \omega(n) \) is the number of distinct prime factors of a square-free number \( n \). Moreover, they gave an effective algorithm to compute such a positive integer solution. In particular, if \( n = p \) is a prime, recently, Yuan and Li [29] considered a variant \( y^2 = px(Ax^2 - 2) \) of Cassels’ equation \( y^2 = 3x(x^2 + 2) \), they proved that the equation has at most five solutions in positive integers \( (x, y) \). Togbé and Yuan [24] improved this result by showing that for any prime \( p \) and any odd positive integer \( A \), the Diophantine equation \( y^2 = px(Ax^2 - 2) \) has at most three solutions in positive integers \( (x, y) \). In [6], Chen proved that the equation \( y^2 = px(Ax^2 + 2) \) has no positive integer solution if \( p \equiv 5, 7 \pmod{8} \), at most one solution if \( p \equiv 1 \pmod{8} \), and at most two solutions if \( p \equiv 3 \pmod{8} \). In [3]-[4], Bennett studied the equation

\[ y^2 = nx(x + 1)(x + 2) \]  

(4)

and obtained results similar to those in [18]. Recently, Zhu and Chen [32] used results on binary quartic equations \( Ax^2 - By^4 = \pm 1 \) by Ljunggren, Cohn, Bennett, and Walsh and Chen et al. (see [13-19]) to prove that each of the equations

\[ y^2 = nx(x^2 \pm 1) \]  

(5)

has at most \( 2^{\omega(n)} - 1 \) solutions \( (x, y) \) and there is an efficient algorithm to compute all their positive integer solutions.

In [7], Chen considered the following Diophantine equation

\[ y^2 = nx(x^2 \pm 1), \]  

(6)

where \( n > 1 \) is a squarefree positive number. He improved Zhu-Chen’s result (see [32]) by proving the following.
Theorem 1.1 For any squarefree positive number \( n \), equation 6 has at most \( 2^{\omega(n)} \) solutions, where \( \omega(n) \) is the number of distinct prime divisors of \( n \).

The first aim of this paper is to extend Theorem 1.1 by obtaining a bound of the number of solutions to the Diophantine equations

\[
y^2 = nx(Ax^2 + 1); \tag{7}
\]

\[
y^2 = nx(Ax^2 - 1). \tag{8}
\]

First, we take \( n = p \) a prime number or \( n = 1 \) and we will prove the following results.

Theorem 1.2 For any prime \( n = p \) and any nonsquare positive integer \( A > 1 \), then each of Diophantine equations 7 and 8 has at most three positive integer solutions \((x, y)\). Moreover, if \( n = 1 \) then each of Diophantine equations 7 and 8 has at most two positive integer solutions \((x, y)\). Moreover, we obtain the following more general result.

Theorem 1.3 For any integer \( n > 1 \) and any nonsquare positive integer \( A > 1 \), Diophantine equations 7 and 8 have at most \( 2^{\omega(n)} + 1 \) and \( 2^{\omega(n)+1} - 1 \) positive integer solutions \((x, y)\), respectively.

In [6], Chen considered the following extension of Cassels’ equation

\[
y^2 = px(x^2 + 2), \tag{9}
\]

where \( p > 3 \) is a prime number. He proved the following result.

Theorem 1.4 For any prime \( p \), equation 9 has no solutions when \( p \equiv 5, 7 \pmod{8} \), at most one positive solution \((x, y)\) when \( p \equiv 1 \pmod{8} \), and at most two such solutions when \( p \equiv 3 \pmod{8} \).

This result was extended by the second author [25] who showed that the equation

\[
y^2 = px(Ax^2 + 2), \tag{10}
\]

where \( p \) is a prime and \( A \) is an odd positive integer, has at most seven positive solutions \((x, y)\).

The second aim of this paper is to extend again Theorem ?? by obtaining a bound of the number of solutions to the Diophantine equations

\[
y^2 = nx(Ax^2 + 2) \tag{11}
\]

and

\[
y^2 = nx(Ax^2 - 2), \tag{12}
\]

where \( n \) is a square-free positive integer, \( A \) is an odd positive integer. So we will prove the following results.
\textbf{Theorem 1.5} For any square-free positive integer $n$ and any odd positive integer $A > 1$, Diophantine equation 11 has at most $3 \cdot 2^{\omega(x)} + 1$ positive integer solutions $(x, y)$.

\textbf{Theorem 1.6} For any square-free positive integer $n$ and any odd positive integer $A > 1$, Diophantine equation 12 has at most $3 \cdot 2^{\omega(x)} - 1$ positive integer solutions $(x, y)$.

In the last part of this paper, we consider the equations

$$y^2 = nx(Ax^2 + 4)$$  \hspace{1cm} (13)

and

$$y^2 = nx(Ax^2 - 4),$$  \hspace{1cm} (14)

where $n$ is a square-free positive integer and $A$ is an odd positive integers.

\textbf{Theorem 1.7} Let $n$ be a square-free positive integer and $A > 1$ an odd positive integers. Then equation 13 has at most $2^{\omega(n)} + 2$ solutions $(x, y)$ when $2 | n$ and at most $3 \cdot 2^{\omega(n)} + 1$ solutions $(x, y)$ when $2 \nmid n$.

\textbf{Theorem 1.8} Let $n$ be a square-free positive integer and $A > 1$ an odd positive integers. Then equation 14 has at most $2^{\omega(n)} + 2$ solutions $(x, y)$.

From Theorem 1.7 and Theorem 1.8, we get

\textbf{Corollary 1.9} If $p$ is an odd prime, then the equation $y^2 = px(Ax^2 + 4)$ has at most 7 solutions $(x, y)$ and the equation $y^2 = px(Ax^2 - 4)$ has at most 8 solutions $(x, y)$.

The organization of this paper is as follows. In Section 2, we recall and prove some results useful for the proofs of our main theorems.

Ljuggren [16] obtained the following result.

\section*{2 Preliminaries}

In this section, we recall and prove some results useful for the proofs of our main theorems.

Ljuggren [16] obtained the following result.
Lemma 2.1 Let $a > 1$ and $b$ be two positive integers. The equation

$$aX^2 - bY^4 = 1$$

has at most one solution in positive integers $X$ and $Y$.

Let $D$ be a positive nonsquare integer and $\epsilon_D = T_1 + U_1 \sqrt{D}$ denote the minimal unit greater than 1, of norm 1, in $\mathbb{Z}[\sqrt{D}]$, and for $k \geq 1$, $\epsilon_D^k = T_k + U_k \sqrt{D}$.

Improving another result of Ljunggren, the second author, Voutier, and Walsh [23] proved the following result.

Lemma 2.2 Let $D$ be a positive nonsquare integer.

1. There are at most two positive integer solutions $(X, Y)$ to equation $X^2 - DY^4 = 1$. If two solutions such that $Y_1 < Y_2$ exist, then $Y_2^4 = T_1 + U_1 \sqrt{D}$, except only if $D = 1785$ or $D = 16 \cdot 1785$, in which case $Y_1^4 = T_1 + U_1 \sqrt{D}$.

2. If only one positive integer solution $(X, Y)$ to the equation $X^2 - DY^4 = 1$ exists, then $Y_1^4 = U_1 \sqrt{D}$, where $U_1 = l\sqrt{2}$ for some squarefree integer $l$, and either $l = 1$, $l = 2$, or $l = p$ for some prime $p \equiv 3 \pmod{4}$.

Chen-Voutier [8] and Yuan [28] independently proved the following result.

Lemma 2.3 Let $d > 2$ be a squarefree integer such that the Pell equation $X^2 - dY^2 = -1$ is solvable in positive integers and let $\tau = v + u\sqrt{d}$ denote its fundamental solution. The only possible integer solution to the equation $X^2 - dY^4 = -1$ is $(X, Y) = (v, \sqrt{u})$.

The upper bound of the number of solutions to equation $aX^4 - bY^2 = 1$ is given by Akhtari [1] and Yuan-Zhang [31], independently. They showed the following result.

Lemma 2.4 The Diophantine equation $aX^4 - bY^2 = 1$ have at most two positive integer solutions $(X, Y)$.

In [16], Ljunggren proved the following result.

Lemma 2.5 If $a$ and $b$ are odd positive integers, then the equation

$$aX^2 - bY^4 = 2$$

(15)

has at most two solutions in positive integers $(X, Y)$.

Assume that $a$ and $b$ are odd positive integer for which the equation

$$aX^2 - bY^2 = 2$$

(16)
is solvable in positive integers \((X,Y)\). Let \((a_1, b_1)\) be the minimal positive solution of equation (16) and define

\[
\alpha = \frac{a_1 \sqrt{a} + b_1 \sqrt{b}}{\sqrt{2}}. \tag{17}
\]

Let \((a_1, b_1)\) be the minimal positive solution \(aX^2 - bY^2 = 2\) such that, for an odd integer \(k\),

\[
\alpha^k = \frac{a_k \sqrt{a} + b_k \sqrt{b}}{\sqrt{2}}, \tag{18}
\]

where \(a_k, b_k\) are positive integers. Luca and Walsh [19] gave more precision on Lemma 2.5.

**Lemma 2.6**

1. If \(b_1\) is not a square, then equation (15) has no solutions.

2. If \(b_1\) is a square and \(b_3\) is not a square, then \((X, Y) = (a_1, \sqrt{b_1})\) is the only solution of equation (15).

3. If \(b_1\) and \(b_3\) are both squares, then \((X, Y) = (a_1, \sqrt{b_1})\) and \((a_3, \sqrt{b_3})\) are the only solutions of equation (15).

The particular case \(b = 1\) was studied by Chen (see Lemma 6 of [6]). The result is the following.

**Lemma 2.7**

If \(b = 1\), the equation (15) has at most one solution \((X, Y)\).

Now, we give the following result of Ljunggren [16].

**Lemma 2.8** [16]

Let \(A, B\) be square-free positive integers, \(A > 1\), the equation \(Ax^2 - By^2 = 1\) has positive integer solutions. Assume that the minimal solution of the equation \(Ax^2 - By^2 = 1\) is \(\tau = x_1 \sqrt{A} + y_1 \sqrt{B}\), then all positive integer solutions of this equation can be expressed as

\[
x_{2k+1} \sqrt{A} + y_{2k+1} \sqrt{B} = (x_1 \sqrt{A} + y_1 \sqrt{B})^{2k+1}, k \geq 0.
\]

Let \(y_1 = lh^2\), \(l\) be square-free, then the quartic Diophantine equation

\[
AX^2 - BY^4 = 1 \tag{19}
\]

only when \(2 \nmid l\) there is positive integer solutions \((X, Y)\), and if \(2 \nmid l\), the only possible solution of the equation 19 is \((X, Y) = (x_l, \sqrt{y_l})\).
Chen-Voutier [8] and Yuan [28] independently proved the following result.

**Lemma 2.9** Let $d > 2$ be a squarefree integer such that the Pell equation $X^2 - dY^2 = -1$ is solvable in positive integers and let $\tau = v + u\sqrt{d}$ denote its fundamental solution. The only possible integer solution to the equation $X^2 - dY^4 = -1$ is $(X, Y) = (v, \sqrt{u})$.

Yuan and Li [30] showed the following result.

**Lemma 2.10** The Diophantine equation $aX^4 - bY^2 = 2$ have at most one positive integer solutions $(X, Y)$.

The two following lemmas give us the bounds on the number of the solutions of the equations

$$AX^4 - BY^2 = 4, \quad (20)$$

and

$$AX^2 - BY^4 = 4. \quad (21)$$

**Lemma 2.11** (Ljunggren [17]) Equation (20) has at most two solutions.

(1) If $x_1 = a^2$ and $Ax_1^2 - 3 = b^2$, where $a, b$ are integers, then equation (20) exactly has two positive integer solutions $X = \sqrt{x_1} = a$ and $X = \sqrt{x_3} = ab$.

(2) If $x_1 = a^2$ and $Ax_1^2 - 3 \neq b^2$, where $a, b$ are integer, then equation 20 has only the positive integer solution $X = \sqrt{x_1} = a$.

(3) If $x_1 = 5a^2$ and $A^2x_1^4 - 5Ax_1^2 + 5 = 5b^2$, where $a, b$ are integers, then equation (20) has only the positive integer solution $X = \sqrt{x_5} = 5ab$.

(4) In other cases, the equation (20) has no positive integer solution.

**Lemma 2.12** (Luo-Yuan [20]) (1) If $y_1$ is not a square, then except $y_1 = 3a^2$ and $By_1 + 3 = 3b^2$, where $a, b$ are integers, equation (21) has only one positive integer solution $(X, Y) = (x_3, \sqrt{y_3})$. In other case, equation (21) has no positive integer solution.

(2) If $y_1$ is a square, then equation (21) except $(X, Y) = (x_1, \sqrt{y_1})$, has most other one positive integer solution $(X, Y) = (x_2, \sqrt{y_2})$, or $(X, Y) = (x_3, \sqrt{y_3})$. If the equation has solution $(X, Y) = (x_3, \sqrt{y_3})$ if and only if $x_1, y_1$ are all square, and $A = 1, B \neq 5$.  


3 Proofs of Theorems 1.2 and 1.3

3.1 Proof of Theorem 1.2 for $y^2 = px(Ax^2 + 1)$

Let $p$ be an odd prime, $A$ a positive integer. Moreover, let $x, y$ be positive integers verifying

$$y^2 = px(Ax^2 + 1).$$

We consider two cases according to the parity of $x$.

**Case 1: $x$ is even.** Then $y$ is also even. Put $x = 2x_1$ and $y = 2y_1$. Therefore, we have

$$2y_1^2 = px_1(4Ax_1^2 + 1).$$

Since $p$ is a prime, then $x_1$ is even, i.e. $x_1 = 2x_2$ (so $4|x$). We deduce that

$$y_1^2 = px_2(16Ax_2^2 + 1).$$

Put $y_1 = pw$. Thus, we have

$$pw^2 = x_2(16Ax_2^2 + 1).$$

Hence, there are positive integers $u, v$ such that

$$x_2 = pu^2, \quad 16Ax_2^2 + 1 = v^2,$$

i.e.

$$v^2 - Ap^2(2u)^4 = 1,$$  \hspace{1cm} (22)

or

$$x_2 = u^2, \quad 16Ax_2^2 + 1 = pv^2,$$

so

$$pv^2 - A(2u)^4 = 1.$$  \hspace{1cm} (23)

We will discuss equations (22) and (23) later.

**Case 2: $x$ is odd.** As $p$ is a prime number, taking $y = pW$, from equation (7) we obtain

$$pW^2 = x(Ax^2 + 1).$$

Then, there exist integers $U, V$ such that

$$x = pU^2, \quad Ax^2 + 1 = V^2,$$

i.e.

$$V^2 - Ap^2U^4 = 1,$$  \hspace{1cm} (24)
or

\[ x = U^2, \quad Ax^2 + 1 = pV^2, \]

so

\[ pV^2 - AU^4 = 1. \]  \hspace{1cm} (25)

Because both equations (22) and (24) are of the type \( X^2 - Ap^2Y^4 = 1 \), by Lemma 2.2, there are at most two solutions satisfying the equations. Notice that \( 2u \neq U \) by \( 2 \not| U \). Hence, there are at most two solutions to (22) and (24) simultaneously. By Lemma 2.1 and as \( p > 1 \), there is at most one solution to (23) and (25) simultaneously as \( 2u \neq U \) and \( x = U^2 \) is odd in (25).

### 3.2 Proof of Theorem 1.2 for \( y^2 = px(Ax^2 - 1) \)

Here we use a method similar to that of Subsection .

**Case 1: \( x \) is even.** Equation (22) becomes

\[ Ap^2(2u)^4 - v^2 = 1, \]  \hspace{1cm} (26)

for \( u, v \) positive integers. In the same way, the corresponding equation to equation (23) is

\[ A(2u)^4 - pv^2 = 1, \]  \hspace{1cm} (27)

for \( u, v \) positive integers.

**Case 2: \( x \) is odd.** Equation (22) implies

\[ Ap^2U^4 - V^2 = 1, \]  \hspace{1cm} (28)

for \( U, V \) positive integers with \( U \) odd. Again, the corresponding equation to equation (23) is

\[ AU^4 - pV^2 = 1, \]  \hspace{1cm} (29)

for \( U, V \) positive integers with \( U \) odd.

Notice that \( U \) is odd and \( 2u \) is even. Lemma 2.4 implies that equations (26) and (28) have at most one positive integer solutions. We also get from Lemma 2.4 that equations (27) and (29) have at most two positive integer solutions.

### 3.3 Proof of Theorem 1.2 for \( y^2 = x(Ax^2 \pm 1) \)

One can reduce equations

\[ y^2 = x(Ax^2 \pm 1) \]  \hspace{1cm} (30)

into equations

\[ v^2 - Au^4 = -1 \]  \hspace{1cm} (31)

and

\[ V^2 - AU^4 = 1, \]  \hspace{1cm} (32)
respectively, where \( u, v, U, V \) are positive integers. By Lemma 2.2, equation (32) has at most two positive integer solutions. By Lemma 2.3, if equation (31) has an integer solution, then it has at most one. This completes the proof of Theorem 1.2.

3.4 Proof of Theorem 1.3

If \( n = n'm^2, \ 1 < m, n' \in \mathbb{N} \), we can rewrite (7) and (8) to \( y^2 = n'x(Ax^2 \pm 1) \) by the mapping \( y/m \mapsto y' \). So without loss of generality, we will assume that \( n \) is square-free. This leads us to get \( y = nW \) and \( x(Ax^2 \pm 1) = nW^2 \).

For each divisor \( n_1 \) of \( n \), let \( n_2 = n/n_1 \). We obtain the factorization \( x = n_1U^2, \ Ax^2 \pm 1 = n_2V^2 \) giving
\[ An_1^2U^4 - n_2V^2 = \mp 1, \]
where \( U, V \) are positive integers.

When \( n_2 = 1 \), by Lemma 2.2 and Lemma 2.3, there are at most two and one positive integer solutions to the equations with negative and positive case, respectively. We know that a square-free integer \( n \) has \( 2^{\omega(n)} \) divisors \( d \) and there are exactly \( 2^{\omega(n)} - 1 \) divisors \( d > 1 \). Therefore, using Lemma 2.1 one can see that each equation
\[ An_1^2U^4 - n_2V^2 = -1 \]
has at most one positive integer solution and there are at most two positive integer solutions to equation
\[ An_1^2U^4 - n_2V^2 = 1 \]
according to Lemma 2.4. Therefore, there are at most \( 2^{\omega(n)} - 1 + 2 \) and \( 2 \cdot (2^{\omega(n)} - 1) + 1 \) positive integer solution solutions to equation (7) and (8), respectively.

Finally, for any given \( n_1 \neq n \) (this implies \( n_2 \neq 1 \)), one of the two Pellian equations \( An_1^2X^2 - n_2Y^2 = 1 \) and \( An_1^2X^2 - n_2Y^2 = -1 \) has no integer solution. This completes the proof of Theorem 1.3.

4 Proofs of Theorems 1.5 and 1.6

Let \( n \) be a positive square-free integer, \( A \) an odd positive integer. Moreover, let \( x, y \) be positive integers verifying
\[ y^2 = nx(Ax^2 + 2). \]

We consider two cases according to the parity of \( x \).
Case 1: $x$ is even. Then $\gcd(x, Ax^2 + 2) = 2$. Put $x = 2z$. Since $n$ is square-free, we put $n = n_1n_2$, where $\gcd(n_1, n_2) = 1$, then $y = 2n_1n_2w$, and we have

$$n_1n_2w^2 = z(2Az^2 + 1).$$

Hence, there are positive integers $u, v$ such that

$$z = n_1u^2, \quad 2Az^2 + 1 = n_2v^2,$$

i.e.

$$n_2v^2 - 2An_1^2u^4 = 1. \quad (35)$$

As $A > 1$, then if $n_2 > 1$, using Lemma 2.1, one can see that equation (35) has at most one positive integer solution $(u, v)$. If $n_2 = 1$, By Lemma 2.2, equation (35) has at most two positive integer solutions. Since the number of factors $d$ of square-free $n$ is $2^{\omega(n)}$, so the equation $y^2 = nx(Ax^2 + 2)$ has at most $(2^{\omega(x)} - 1) + 2 = 2^{\omega(x)} + 1$ positive integer solutions.

Case 2: $x$ is odd. As $n$ is square-free, taking $w = \frac{y}{n_1n_2}$, one can see that equation (11) implies

$$n_1n_2w^2 = x(Ax^2 + 2).$$

Then, there exist odd integers $u, v$ such that

$$x = n_1u^2, \quad Ax^2 + 2 = n_2v^2,$$

i.e.

$$n_2v^2 - An_1^2u^4 = 2. \quad (36)$$

By Lemma 2.5 or Lemma 2.6, equation (37) has at most two positive integer solutions $(u, v)$. So the equation $y^2 = nx(Ax^2 + 2)$ has at most $2 \cdot 2^{\omega(x)} = 2^{\omega(x) + 1}$ positive integer solutions $(u, v)$. This completes the proof of Theorem 1.5.

If $x, y$ are positive integers verifying

$$y^2 = nx(Ax^2 - 2),$$

we use a method similar to that of the proof of Theorem 1.5.

In equations (35) and (36), 1 becomes $-1$, 2 becomes $-2$, we deduce

$$n_2v^2 - 2An_1^2u^4 = -1, \quad (37)$$

and

$$n_2v^2 - An_1^2u^4 = 2. \quad (38)$$

Therefore, by Lemma 2.3, 2.4, and Lemma 2.10, one can see that equation (37) has at most one (if $n_2 = 1$), two (if $n_2 > 1$) positive integer solutions and equation (38) has at most two positive integer solutions. So equation (12) has at most $2 \cdot (2^{\omega(x)} - 1) + 1 + 2^{\omega(x)} = 3 \cdot 2^{\omega(x)} - 1$ positive integer solutions $(x, y)$. This completes the proof of Theorem 1.6.
5 Proofs of Theorems 1.7 and 1.8

In this section, we will prove our last two results. First, we study equation (13).

5.1 Proof of Theorem 1.7.

Since \( \gcd(x, Ax^2 + 4) = 1 \), then from equation (13), we have

\[
x = n_1u^2, \quad Ax^2 + 4 = n_2v^2, \quad n = n_1n_2, \quad y = n_1n_2uv, \quad (39)
\]

or

\[
x = 2n_1u^2, \quad Ax^2 + 4 = 2n_2v^2, \quad n = n_1n_2, \quad y = 2n_1n_2uv, \quad (40)
\]

where \( n_1, n_2, u, v \) are all positive integers.

From equation (39), we get

\[
n_2v^2 - An_1^2u^4 = 4. \quad (41)
\]

If \( 2|n_1, 2 \nmid n_2 \), then equation (41) becomes

\[
n_2v^2 - An_1^2u^4 = 1, \quad n_1 = 2n'_1, \quad v = 2v'.
\]

If \( 2 \nmid n_1, 2|n_2 \), note that \( 2 \nmid A \) and \( n \) are square-free numbers. Thus, from (41) we get \( 2|u \) and then \( 2|v \). Therefore, equation (41) has no integer solution. If \( n_2 > 1 \), then Lemma 2.12 implies that equation (41) has at most one positive integer solution \((u, v')\). If \( n_2 = 1 \), from Lemma 2.12, we see that equation (41) has at most two positive integer solutions \((u, v')\).

When \( 2 \nmid n_1 \) and \( 2 \nmid n_2 \), if \( 2|u \), then \( 2|v \), so equation (41) becomes

\[
n_2v^2 - An_1^2u^4 = 1, \quad v = 2v', \quad u = 2u'.
\]

Similarly, using Lemma 2.12 and Lemma 2.2, one can see that if the above equation has at most one positive integer solution \((u', v')\) when \( n_2 > 1 \) and has at most two positive integer solutions \((u', v')\) when \( n_2 = 1 \). Moreover, the two above equations cannot simultaneously have one (or two) solutions. Hence, in equation (41), we need to consider the case \( 2 \nmid u, 2 \nmid v \). As \( 2 \nmid A \), the equation \( n_2x^2 - Ay^2 = 4 \) has an odd solution. From Lemma 2.12, we see that equation (41) has at most one positive integer solutions \((u, v)\). Thus, equation (41) has at most three positive integer solutions \((u, v)\) when \( n_2 = 1 \) and at most two positive integer solutions \((u, v)\) when \( n_2 > 1 \).

Now let us study equation (40) which implies

\[
n_2v^2 - 2An_1^2u^4 = 2. \quad (42)
\]
Thus, $2|(n_2v^2)$. From (42) we deduce
\[ n'_2v^2 - An'_1u^4 = 1, n_2 = 2n'_2, \]
or
\[ 2n_2v'^2 - An^2u^4 = 1, v = 2v', 2 \nmid n_2. \]
From Lemma 2.12 and Lemma 2.2, the two above equations have at most one positive integer solution $(u, v)$ or $(u, v')$ when $n'_2 > 1$ and the first equation has at most two positive integer solutions $(u, v)$ when $n'_2 = 1$.

Since the number of factors $d$ of square-free $n$ is $2^{\omega(n)}$ so there are $2^{\omega(n)} - 1$ factors $d > 1$, and if $2|n$, there are $2^{\omega(n)-1}$ factors with $2 \nmid d$. From equation (13) we get equation (41) or (42). Therefore, if $2|n$, equation (13) gives $n_2v^2 - An_1u^4 = 1$ or $n'_2v^2 - An'_1u^4 = 1$. Thus equation (13) has at most $2((2^{\omega(n)} - 1) + 2) = 2^{\omega(n)} + 2$ positive integer solutions $(x, y)$. Similarly, if $2 \nmid n$, equation (13) has at most $2(2^{\omega(n)} - 1) + 3 + 2^{\omega(n)} = 3 \cdot 2^{\omega(n)} + 1$ positive integer solutions $(x, y)$. So the proof of Theorem 1.7 is complete.

5.2 Proof of Theorem 1.8

Using the same method, we obtain a similar conclusion for equation (14).

Since $\gcd(x, Ax^2 - 4) = 1, 2, 4$, then from equation (14), we have
\[ n_2v^2 - An^2u^4 = -4. \]  
\[ n_2v^2 - 2An^2u^4 = -2, \]
where $n_1, n_2, u, v$ are all positive integers. Using a discussion similar to that of Theorem 1.7, with Lemmas 2.12, 2.2, 2.6, 2.12 replaced by Lemmas 2.2, 2.9, 2.10, 2.11, ??, one can complete the proof of Theorem 1.8. ??.

5.3 Examples

In this subsection, we give some examples to show how to solve equations (13) and (14). Using the algorithms deduced from the proofs of Theorems 1.7 and 1.8, we can obtain all solutions of equations (13) and (14) when $A = 1$ and $n < 60$. We have

**Corollary 5.1** If $n$ has no square factor and $1 < n < 60$, then

1. The solutions of the equation $y^2 = nx(x^2 + 4)$ are $(n, x, y) = (5, 1, 5), (5, 4, 20), (7, 14, 140), (13, 36, 780), (15, 6, 60), (34, 8, 136), (39, 3, 39), (41, 18, 492), (41, 64, 3280), (41, 82, 4756), (55, 11, 275)$.

2. The solutions of the equation $y^2 = nx(x^2 - 4)$ are $(n, x, y) = (3, 4, 12), (3, 6, 24), (3, 98, 1680), (7, 16, 168), (10, 18, 240), (11, 198, 1240), (15, 3, 15),...
(15, 10, 120), (17, 34, 816), (30, 8, 120), (35, 7, 105), (39, 52, 2340), (42, 14, 336), (51, 100, 7140), (55, 20, 660), (58, 19602, 20900880).

**Proof:** For every $n$, using the proofs of Theorems 1.7, 1.8, and Lemma 2.1 - Lemma 2.6, we can prove Corollary 1.9.

For example, when $A = 1$ and $n = 41$, from equation (13) we get $41v^2 - u^4 = 4$, or $v^2 - 2 \cdot 41^2u^4 = 2$, or $41v^2 - 2u^4 = 2$. In equation $41v^2 - u^4 = 4$, if $u, v$ are all even, then the equation becomes $41(v^2)^2 - 4(y^2)^4 = 1$. Since the minimum solution of $41x^2 - y^2 = 1$ is $x_1 = 5, y_1 = 32$, so by Lemma 2.12, we have the equation $41v^2 - u^4 = 4$ has only the solution $(u, v) = (8, 10)$. If $u, v$ are all odd, because all positive integer solutions of $41x^2 - y^2 = 4$ was give by $y + x \sqrt{41} = (32 + 5y)k$, where $k \geq 1$ is an integer, then the equation has no odd solution. Hence, the equation $41v^2 - u^4 = 4$ has no solution. Similarly, by Lemma 2.12, the equations $v^2 - 2 \cdot 41^2u^4 = 2$ and $41v^2 - 2u^4 = 2$ have only the solutions $(u, v) = (1, 58), (3, 2)$, respectively. Thus, the equation $y^2 = 41x(x^2 + 4)$ has only the positive integer solutions are $(x, y) = (64, 3280), (82, 4756), (18, 492)$.

When $A = 1$ and $n = 58$, from equation (14) we get $u^4 - 58v^2 = 4$, or $4u^4 - 29v^2 = 4$, or $29^2u^4 - 2v^2 = 4$, or $58^2u^4 - v^2 = 4$, or $8u^4 - 29v^2 = 2$, or $29^2u^4 - v^2 = 1$, or $8 \cdot 29^2u^4 - v^2 = 2$. It is clear that the above equations have no solutions except the second equation $u^4 - 29v^2 = 1$. Since the minimum solution of the Pell equation $x^2 - 29y^2 = 1$ is $x_1 = 1820, y_1 = 9801 = 99^2$, then from Lemma 2.12, we see that the equation $u^4 - 29v^2 = 1$ has only the positive integer solution $(u, v') = (99, 1820)$. Thus the equation $y^2 = 58x(x^2 - 4)$ has only the positive integer solution $(x, y) = (19602, 20900880)$.

**Acknowledgements**

A. T. is partially supported by Purdue University North Central. The other authors are supported by the Natural Science Research Fund of Sichuan Provincial Department of Education (No.2011JY0031, No.12ZB002).

**References**


On a family of permutation polynomials of $\mathbb{F}_q$

Kacem Belghaba
Laboratoire de Mathématiques et ses Applications
Université d’Oran, BP 1524, Algeria
E-mail: belghaba.kacem@univ-oran.dz

Salima Kebli
Laboratoire de Mathématiques et ses Applications
Université d’Oran, BP 1524, Algeria
E-mail: smkebli06@sci.just.edu.jo

Abstract

We construct a new family of permutation polynomials over $\mathbb{F}_q$.

2000 Mathematics Subject Classification: 11T06, 12E20.
Keywords: Finite fields, permutation polynomials.

1 Introduction

Let $\mathbb{F}_q$ be the finite field of characteristic $p$ containing $q = p^r$ elements. A polynomial $f(x) \in \mathbb{F}_q[x]$ is called a permutation polynomial of $\mathbb{F}_q$ if the induced map $f : \mathbb{F}_q \to \mathbb{F}_q$ is one-to-one. The study of permutation polynomials was initiated by Hermite [3] for $\mathbb{F}_p$ and then generalized by Dickson [2] to $\mathbb{F}_q$. The interest on permutation polynomials increased in part because of their application in cryptography and coding theory. We refer to [5] or [7] for the basic results on permutation polynomials. One of the open problems proposed by Lidl and Mullen [4], is to find new classes of permutation polynomials of $\mathbb{F}_q$. Despite the interest of numerous people on the subject, characterizing permutation polynomials and finding new families of permutation polynomials remain open questions. Ayad and Kihel [1] used only elementary techniques to give sufficient conditions on $u$ and $q$ such that the polynomial $x^u(1 + x^{q^{1/2}} + x^{q^{1/4}})$ represents a permutation polynomial over $\mathbb{F}_q$.

In this paper, we will extend Theorem 1 of [1] to the family of polynomials $h(x) = x^u(1 \pm x^{q^{1/4}} \pm x^{q^{1/2}})$ and give necessary conditions such that $h(x)$ is a permutation polynomial of $\mathbb{F}_q$. Our proofs rest heavily on the application of
the following theorem proved by Wan and Lidl [10].

1 Theorem: Let \( g \) be a primitive element of \( \mathbb{F}_q \) and let \( w = g^{q^{-1}} \) be a primitive \( l \)-root of unity in \( \mathbb{F}_q \). Then the polynomial \( h(x) = x^u \left( f \left( x^\left( \frac{q-1}{4} \right) \right) \right) \) in a permutation of \( \mathbb{F}_q \) if and only if the following conditions hold:

(a) \( (u, \frac{q-1}{4}) = 1 \).

(b) \( f(w^t) \neq 0 \), for each \( t = 0, 1, \ldots, l-1 \).

(c) For all \( 0 \leq i < j < l \),

\[
\text{Ind}_g \left( \frac{f(w^i)}{f(w^j)} \right) \neq u(j-i) \mod l,
\]

where \( \text{Ind}_g \left( \frac{f(w^i)}{f(w^j)} \right) \) is the residue class \( b \) module \( (q-1) \) such that \( \frac{f(w^i)}{f(w^j)} = g^b \).

2 Family of permutation polynomials of \( \mathbb{F}_q \)

Let \( \mathbb{F}_q \) be the finite field of characteristic \( p \) containing \( q = p^r \) elements. A polynomial \( f(x) \in \mathbb{F}_q[x] \) is called a permutation polynomial of \( \mathbb{F}_q \) if the induced map \( f : \mathbb{F}_q \to \mathbb{F}_q \) is one-to-one. In the first part of this section, we construct a new family of permutation polynomials over \( \mathbb{F}_q \). We prove the following.

Theorem 2 Let \( p \neq 5 \) be a prime number, and \( q = p^r \), where \( r \) is a positive integer such that \( q \equiv 1 \mod 4 \). Let \( g \) be a primitive element in \( \mathbb{F}_q \) and \( \omega = g^{\frac{q-1}{4}} \) be a primitive 4-root of unity in \( \mathbb{F}_q \). Then the polynomial

\[
h(x) = x^u \left( 1 + x^{\frac{q-1}{4}} - x^{\frac{q+1}{2}} \right)
\]

is a permutation polynomial of \( \mathbb{F}_q \) if and only if one the following conditions holds.

(1) \( q \equiv 1 \mod 8 \), \( \gcd(u, \frac{q-1}{4}) = 1 \) and \( (2 + \omega)^{\frac{q-1}{4}} = \pm 1 \)

(2) \( q \not\equiv 1 \mod 8 \), \( \gcd(u, \frac{q-1}{4}) = 1 \), \( u \) even and \( (2 + \omega)^{\frac{q-1}{4}} \neq \pm 1 \).

Proof:
Suppose that

\[
h(x) = x^u \left( 1 + x^{\frac{q-1}{4}} - x^{\frac{q+1}{2}} \right)
\]

is a permutation polynomial of \( \mathbb{F}_q \), where \( q \equiv 1 \mod 4 \). Then Theorem 1 implies that \( \gcd(u, \frac{q-1}{4}) = 1 \). Let

\[
f(x) = -x^2 + x + 1.
\]
Then
\[ h(x) = x^u f\left(\frac{x^{q-1}}{4}\right). \]

We have the following two cases to discuss.

Case 1: \( q \equiv 1 \mod 8. \)

Then Theorem 1.(c) implies that
\[ \text{Ind}_g\left(\frac{f(\omega^i)}{f(\omega^j)}\right) \neq u(j - i) \mod 4. \]

Hence we have the following.

- If \( i = 0, j = 1, \) then
  \[ \frac{f(\omega^i)}{f(\omega^j)} = g^e \]
  in \( \mathbb{F}_q, \) where
  \[ e = \text{Ind}_g\left(\frac{f(\omega^i)}{f(\omega^j)}\right). \]
  Then \( \frac{1}{\omega^{\frac{q-1}{4}}} = g^e, \) whereupon
  \[ (2 + \omega)^{\frac{q-1}{4}}\omega^e = 1. \]
  Theorem 1.(c) implies that \( e \not\equiv u \mod 4. \) But \( \gcd(u, \frac{q-1}{4}) = 1 \) implies that \( u \) is odd.

- If \( i = 0, j = 2, \) then
  \[ \frac{f(\omega^j)}{f(\omega^j)} = g^e, \]
  whereupon
  \[ (-1)^{\frac{q-1}{4}} = \omega^e, \]
  i.e. \( \omega^e = 1. \) Hence \( e \equiv 0 \mod 4. \) Theorem 1.(c) implies that \( e \not\equiv 2u \mod 4. \)

- If \( i = 0, j = 3, \) then
  \[ \frac{f(\omega^j)}{f(\omega^j)} = \frac{1}{2 - \omega} = g^e \]
  Hence \( (2 - \omega)^{\frac{q-1}{4}}\omega^e = 1. \) Theorem 1.(c) implies that
  \[ e \not\equiv 3u \mod 4. \]

- If \( i = 1, j = 2, \) then
\[
\frac{f(\omega^i)}{f(\omega^j)} = -(2 + \omega) = g^e.
\]

Hence
\[
(2 + \omega)^\frac{q-1}{4} = \omega^e.
\]

Theorem 1.(c) implies that \( e \not\equiv u \mod 4 \).

- \( i = 1, j = 3 \), then
\[
\frac{f(\omega^i)}{f(\omega^j)} = \frac{2 + \omega}{2 - \omega} = g^e.
\]

Hence
\[
(2 + \omega)^\frac{q-1}{4} = (2 - \omega)^\frac{q-1}{4} \omega^e.
\]

Theorem 1.(c) implies that \( e \not\equiv 2u \mod 4 \).

- \( i = 2, j = 3 \), then
\[
\frac{f(\omega^i)}{f(\omega^j)} = (2 - \omega) = g^e.
\]

Hence
\[
(2 - \omega)^\frac{q-1}{4} = \omega^e.
\]

Theorem 1.(c) implies that \( e \not\equiv u \mod 4 \). From above, \( u \) is odd, then

\[
u \equiv 1 \text{ or } 3 \mod 4.
\]

If \( u \equiv 1 \mod 4 \), then the above equation \((2 + \omega)^\frac{q-1}{4} \omega^e = 1\), with \( e \not\equiv u \mod 4 \), implies that

\[
(2 + \omega)^\frac{q-1}{4} \neq \omega^3.
\]

The above equation \((2 + \omega)^\frac{q-1}{4} = \omega^e\), with \( e \not\equiv u \mod 4 \), implies that

\[
(2 + \omega)^\frac{q-1}{4} \neq \omega.
\]

Hence \((2 + \omega)^\frac{q-1}{4} = \pm 1\).

If \( u \equiv 3 \mod 4 \), the same above equations imply that

\[
(2 + \omega)^\frac{q-1}{4} = \pm 1.
\]

Case 2: \( g \not\equiv 1 \mod 8 \), then similar calculations imply that \( u \) is even and \((2 + \omega)^\frac{q-1}{4} = \pm \omega \).

For the reciprocal, suppose that condition (1) of the theorem is satisfied. Then clearly condition (a) of Theorem 1 is satisfied. In fact, as \( p \neq 5 \) we have

\[
f(\omega^0) = 1 \neq 0,
\]
\[ f(\omega) = 1 + \omega - \omega^2 = 2 + \omega \neq 0, \]
\[ f(\omega^2) = 1 + \omega^2 - \omega^4 = 1 \neq 0, \]
\[ f(\omega^3) = 1 + \omega^3 - \omega^6 = 1 - \omega + 1 = 2 - \omega \neq 0. \]

Then \( f(\omega^i) \neq 0 \) for every \( i \) such that \( 0 \leq i < 4 \).

- If \( i = 0, j = 1 \), then the equation
  \[ \frac{f(\omega^i)}{f(\omega^j)} = \frac{1}{2 + \omega} = g^e \]
  implies that \((2 + \omega)^{\frac{q-1}{4}} = \omega^e\). We have \((2 + \omega)^{\frac{q-1}{4}} = \pm 1\), which implies that \( e \) is even. But \( u \) is odd, then \( e \not\equiv u(j-i) \mod 4 \).

- \( i = 0, j = 2 \), then the equation
  \[ \frac{f(\omega^i)}{f(\omega^j)} = \frac{1}{1 + \omega^2 - \omega^4} = -1 = g^e \]
  implies that \((-1)^{\frac{q-1}{4}} = 1 = \omega^e\). Hence \( e \equiv 0 \mod 4 \). Then \( e \not\equiv 2u \mod 4 \), i.e \( e \not\equiv u(j-i) \mod 4 \).

- \( i = 0, j = 3 \), then the equation
  \[ \frac{f(\omega^i)}{f(\omega^j)} = \frac{1}{1 + \omega^3 - \omega^6} = \frac{1}{2 - \omega} = g^e \]
  implies that \((2 - \omega)^{\frac{q-1}{4}} = \omega^e\). We have \((2 + \omega)^{\frac{q-1}{4}} = \pm 1\), implying that \((2 - \omega)^{\frac{q-1}{4}} = \pm 1\). Hence \( \omega^e = \pm 1 \), whereupon \( e \) is even. Therefore, \( e \not\equiv u(j-i) \mod 4 \).

- \( i = 1, j = 3 \), then the equation
  \[ \frac{f(\omega^i)}{f(\omega^j)} = \frac{1 + \omega - \omega^2}{1 + \omega^3 - \omega^6} = \frac{2 + \omega}{2 - \omega} = g^e, \]
  implies that
  \[ (2 + \omega)^{\frac{q-1}{4}} = (2 - \omega)^{\frac{q-1}{4}} \omega^e \]
  whereupon \( \omega^e = 1 \). Then \( e \equiv 0 \mod 4 \). Hence \( e \not\equiv u(j-i) \mod 4 \).

- \( i = 2, j = 3 \), then the equation
\[
\frac{f(\omega^i)}{f(\omega^j)} = \frac{1 + \omega^2 - \omega^4}{1 + \omega^3 - \omega^6} = -(2 - \omega) = g^e
\]
implies that
\[
(-1)^{\frac{q-1}{4}} = (2 - \omega)\frac{q-1}{4}\omega^e,
\]
i.e \((2 - \omega)^{\frac{q-1}{4}}\omega^e = 1\). But \((2 - \omega)^{\frac{q-1}{4}} = \pm 1\), then \(\omega^e = \pm 1\). Hence, \(e\) is even.

Therefore,
\[e \neq u(j - i) \mod 4.\]

The case \(q \not\equiv 1 \mod 8\) is done similarly. □

Ayad and Kihel [1] proved the following.

**Theorem 3** Let \(u\) be a positive integer and let
\[
h(x) = x^u(x^{\frac{q-1}{2}} + x^{\frac{q-1}{4}} + 1)
\]
Assume that the following conditions hold.

(i) \(\gcd(u, q - 1) = 1\).

(ii) \(q \equiv 1 \pmod{8}\).

(iii) \(3^{\frac{q-1}{4}} \equiv 1 \pmod{p}\).

Then \(h(x)\) is a permutation polynomial of \(\mathbb{F}_q\).

We will extend Theorem 3 and prove the following.

**Theorem 4** Let \(p\) be a prime number, and \(q = p^r\) where \(r\) is a positive integer such that \(q \equiv 1 \pmod{4}\). Let \(h(x) = x^u(x^{\frac{q-1}{2}} + x^{\frac{q-1}{4}} + 1)\) be a polynomial over \(\mathbb{F}_q\). Then \(h\) is a permutation polynomial of \(\mathbb{F}_q\), if and only if one of the following conditions hold.

(1) \(q \equiv 1 \pmod{8}\), \(\gcd(u, \frac{q-1}{4}) = 1\) and \(3^{\frac{q-1}{4}} \equiv 1 \pmod{p}\).

(2) \(q \not\equiv 1 \pmod{8}\), \(\gcd(u, \frac{q-1}{4}) = 1\), \(u\) even and \(3^{\frac{q-1}{4}} \equiv -1 \pmod{p}\).

**Proof:** Suppose that \(q \equiv 1 \pmod{8}\). Let \(f(x) = x^2 + x + 1\) and \(s = \frac{q-1}{4}\). Then \(h(x) = x^u f(x^s)\). If \(h(x)\) is a permutation polynomial over \(\mathbb{F}_q\), then Theorem 1 (c) implies that
\[
\text{Ind}_q \frac{f(u^i)}{f(u^j)} \neq u(j - i) \pmod{4},
\]
for every \(0 \leq i < j < 4\). Let \(\frac{f(u^i)}{f(u^j)} = g^e\), hence we have the following.
• If \( i = 0, j = 1 \), then \( f(u^i) = f(u^j) = \frac{3}{w^2 + w + 1} = \frac{3}{w} \), since \( w^2 = -1 \). Hence \( \frac{3}{w} = g^e \), which implies that

\[
\left( \frac{3}{w} \right)^{\frac{q-1}{4}} = \left( g^e \right)^{\frac{q-1}{4}}.
\]

Then \( \frac{3^{q-1}}{4} = w^{\frac{q-1}{4}} w^e \), whereupon

\[
3^{\frac{q-1}{4}} = \left\{
\begin{array}{ll}
w^e & \text{if } q \equiv 1 \pmod{16} \\
-w^e & \text{if } q \not\equiv 1 \pmod{16},
\end{array}
\right.
\]

while \( e \not\equiv u \pmod{4} \). From Theorem 1 (a), we have \( \gcd(u, \frac{q-1}{4}) = 1 \), implying that \( u \) is odd since \( \frac{q-1}{4} \) is even.

• If \( i = 0, j = 2 \), then \( f(u^i) = f(u^j) = \frac{3}{w^6 + w^3 + 1} = 3 = g^e \). Hence \( \frac{3^{q-1}}{4} = \left( g^e \right)^{\frac{q-1}{4}} = w^e \), with \( e \not\equiv u(2 - 0) \pmod{4} \). Therefore, \( e \not\equiv 2 \pmod{4} \) since \( u \) is odd. It follows that

\[
3^{\frac{q-1}{4}} \not\equiv 1 \pmod{p}.
\]

• If \( i = 0, j = 3 \) then \( f(u^i) = f(u^j) = \frac{3}{w^9 + w^6 + w^3 + 1} = \frac{3}{w^3} = \frac{-3}{w} \). Hence \( \frac{-3}{w} = g^e \), which implies that \( (-3)^{\frac{q-1}{4}} = w^{\frac{q-1}{4}} w^e \), with \( e \not\equiv 3u \pmod{4} \). Then

\[
3^{\frac{q-1}{4}} = \left\{
\begin{array}{ll}
w^e & \text{if } q \equiv 1 \pmod{16} \\
-w^e & \text{if } q \not\equiv 1 \pmod{16},
\end{array}
\right.
\]

where \( e \not\equiv 3u \pmod{4} \).

Then equations (1) and (3) imply that

\[
3^{\frac{q-1}{4}} = \left\{
\begin{array}{ll}
w^e & \text{if } q \equiv 1 \pmod{16} \\
-w^e & \text{if } q \not\equiv 1 \pmod{16},
\end{array}
\right.
\]

where \( e \not\equiv u \pmod{4} \) and \( e \not\equiv 3u \pmod{4} \). From above \( u \) is odd, then \( e \) is even. Hence (4) implies that

\[
3^{\frac{q-1}{4}} \equiv \pm 1 \pmod{p}.
\]

Equation (3) implies that \( 3^{\frac{q-1}{4}} \not\equiv -1 \pmod{p} \), then \( 3^{\frac{q-1}{4}} \equiv 1 \pmod{p} \).

For the reciprocal, suppose that \( q \equiv 1 \pmod{8} \), \( \gcd(u, \frac{q-1}{4}) = 1 \), and \( 3^{\frac{q-1}{4}} \equiv 1 \pmod{p} \).
Kacem Belghaba, Salima Kebli

If $gcd(u, \frac{q-1}{4}) = 1$, then $gcd(u, q - 1) = 1$. Clearly, $3^{\frac{q-1}{4}} = \pm 1$ or $\pm w$. We suppose that $3^{\frac{q-1}{4}} \equiv 1 \mod p$, then Theorem 3 implies the result.

If $q \not\equiv 1 \mod 8$, then $q \equiv 5 \mod 8$. Hence,

$$\frac{q-1}{4} \equiv \begin{cases} 
1 \mod 4 \\
3 \mod 4.
\end{cases}$$

Therefore, we have the following cases:

Case 1: $\frac{q-1}{4} \equiv 1 \mod 4$

- If $i = 1$ and $j = 2$. Then

$$\frac{f(w^i)}{f(w^j)} = \frac{w^2 + w + 1}{w^4 + w^2 + 1} = w = g^e.$$ 

Hence $w^{\frac{q-1}{4}} = w^e$, which implies that $e \equiv \frac{q-1}{4} \mod 4$. Theorem 1 (c) implies that $u \not\equiv 1 \mod 4$.

- If $i = 1$ and $j = 3$. Then

$$\frac{f(w^i)}{f(w^j)} = \frac{w^2 + w + 1}{w^6 + w^3 + 1} = \frac{w}{w^3} = w^2 = g^e.$$ 

Hence, $(-1)^{\frac{q-1}{4}} = w^e$, which implies that $e \equiv 2 \mod 4$. Theorem 1 (c) implies that $e \not\equiv 2u \mod 4$.

whereupon $u \not\equiv 3 \mod 4$. Then $u$ is even. If $u \equiv 2 \mod 4$, we have the following:

- If $i = 0$ and $j = 2$. Then

$$\frac{f(w^i)}{f(w^j)} = \frac{3}{w^4 + w^2 + 1} = 3 = g^e.$$ 

Hence $3^{\frac{q-1}{4}} = w^e$. Theorem 1 (c) implies that $e \not\equiv 2u \equiv 0 \mod 4$. Hence $3^{\frac{q-1}{4}} \not\equiv 1 \mod 4$.

- If $i = 0$ and $j = 1$. Then

$$\frac{f(w^i)}{f(w^j)} = \frac{3}{w^2 + w + 1} = \frac{3}{w} = g^e.$$ 

Hence $3^{\frac{q-1}{4}} = w^{\frac{q-1}{4}}w^e$, i.e. $3^{\frac{q-1}{4}} = w^{e+1}$ Theorem 1 (c) implies that $e \not\equiv u \equiv 2 \mod 4$. Hence $3^{\frac{q-1}{4}} \not\equiv w^3$. 

If $i = 0$ and $j = 3$. Then
\[
\frac{f(w^i)}{f(w^j)} = \frac{3}{w^6 + w^3 + 1} = \frac{3}{w^3} = g^e.
\]
Hence $3^{\frac{q-1}{4}} = w^{3\frac{q-1}{4}} w^e$, i.e. $3^{\frac{q-1}{4}} = w^{e+3}$. Theorem 1 (c) implies that $e \not\equiv 3u \equiv 2 \mod 4$. Hence $3^{\frac{q-1}{4}} \neq w$.

whereupon
\[
3^{\frac{q-1}{4}} \equiv -1 \mod p.
\]
If $u \equiv 0 \mod 4$, we have the following:

- If $i = 0$ and $j = 2$. Then
  \[
  \frac{f(w^i)}{f(w^j)} = \frac{3}{w^4 + w^2 + 1} = 3 = g^e.
  \]
  Hence $3^{\frac{q-1}{4}} = w^e$, Theorem 1 (c) implies that $e \not\equiv 2u \equiv 0 \mod 4$. Then $3^{\frac{q-1}{4}} \neq 1 \mod p$.

- If $i = 0$ and $j = 1$. Then
  \[
  \frac{f(w^i)}{f(w^j)} = \frac{3}{w^2 + w + 1} = \frac{3}{w} = g^e.
  \]
  Hence $3^{\frac{q-1}{4}} = w^{\frac{q-1}{4}} w^e$, i.e. $3^{\frac{q-1}{4}} = w^{e+1} \mod p$. Theorem 1 (c) implies that $e \not\equiv u \equiv 0 \mod 4$. Then $3^{\frac{q-1}{4}} \neq w$.

- If $i = 0$ and $j = 3$. Then
  \[
  \frac{f(w^i)}{f(w^j)} = \frac{3}{w^6 + w^3 + 1} = \frac{3}{w^3} = g^e.
  \]
  Hence $3^{\frac{q-1}{4}} = w^{3\frac{q-1}{4}} w^e$, i.e. $3^{\frac{q-1}{4}} = w^{e+3}$. Theorem 1 (c) implies that $e \not\equiv 3u \equiv 0 \mod 4$. Then $3^{\frac{q-1}{4}} \neq w$.

then
\[
3^{\frac{q-1}{4}} \equiv -1 \mod p.
\]
Case 2: $\frac{q-1}{4} \equiv 3 \mod 4$.

- If $i = 1$ and $j = 2$. Then
  \[
  \frac{f(w^i)}{f(w^j)} = \frac{w^2 + w + 1}{w^4 + w^2 + 1} = w = g^e.
  \]
  Hence $w^{\frac{q-1}{4}} = w^e$, which implies that $e \equiv \frac{q-1}{4} \equiv 3 \mod 4$. Theorem 1 (c) implies that $u \not\equiv 3 \mod 4$. 

• If $i = 1$ and $j = 3$. Then
\[
\frac{f(w^i)}{f(w^j)} = \frac{w^2 + w + 1}{w^6 + w^3 + 1} = \frac{w}{w^3} = w^2 = g^e.
\]

Hence, $(-1)^{\frac{q-1}{2}} = w^e$, which implies that $e \equiv 2 \mod 4$. Theorem 1 (c) implies that $e \not\equiv 2u \mod 4$.

whereupon $u \not\equiv 1 \mod 4$ and $u \not\equiv 3 \mod 4$. Then $u$ is even. If $u \equiv 2 \mod 4$, we have the following:

• If $i = 0$ and $j = 2$. Then
\[
\frac{f(w^i)}{f(w^j)} = \frac{3}{w^4 + w^2 + 1} = \frac{3}{w} = g^e.
\]

Hence $3^{\frac{q-1}{2}} = w^e$, Theorem 1 (c) implies that $e \not\equiv 2u \equiv 0 \mod 4$. Then $3^{\frac{q-1}{2}} \not\equiv 1 \mod 4$.

• If $i = 0$ and $j = 1$. Then
\[
\frac{f(w^i)}{f(w^j)} = \frac{3}{w^2 + w + 1} = \frac{3}{w} = g^e.
\]

Hence $3^{\frac{q-1}{4}} = w^{\frac{q-1}{2}}w^e$, i.e. $3^{\frac{q-1}{4}} = w^{e+1}$ Theorem 1 (c) implies that $e \not\equiv u \equiv 2 \mod 4$. Hence $3^{\frac{q-1}{4}} \not\equiv w^3$.

• If $i = 0$ and $j = 3$. Then
\[
\frac{f(w^i)}{f(w^j)} = \frac{3}{w^6 + w^3 + 1} = \frac{3}{w^3} = g^e.
\]

Hence $3^{\frac{q-1}{4}} = w^{3^{\frac{q-1}{4}}}w^e$, i.e. $3^{\frac{q-1}{4}} = w^{e+3}$ Theorem 1 (c) implies that $e \not\equiv 3u \equiv 2 \mod 4$. Hence $3^{\frac{q-1}{4}} \not\equiv w$.

whereupon
\[
3^{\frac{q-1}{4}} \equiv -1 \pmod{p}.
\]

If $u \equiv 0 \mod 4$, we have the following:

• If $i = 0$ and $j = 2$. Then
\[
\frac{f(w^i)}{f(w^j)} = \frac{3}{w^4 + w^2 + 1} = \frac{3}{w} = g^e.
\]

Hence $3^{\frac{q-1}{4}} = w^e$, Theorem 1 (c) implies that $e \not\equiv 2u \equiv 0 \mod 4$. Then $3^{\frac{q-1}{4}} \not\equiv 1 \pmod{p}$. 
• If $i = 0$ and $j = 1$. Then
\[
\frac{f(w^i)}{f(w^j)} = \frac{3}{w^2 + w + 1} = \frac{3}{w} = g^e.
\]
Hence $3^{2^{-1}} = w^{\frac{e+1}{2}}$, i.e. $3^{2^{-1}} = w^{e+3}$. Theorem 1 (c) implies that $e \not\equiv u \equiv 0 \mod 4$. Then $3^{2^{-1}} \neq w^3$.

• If $i = 0$ and $j = 3$. Then
\[
\frac{f(w^i)}{f(w^j)} = \frac{3}{w^6 + w^3 + 1} = \frac{3}{w^3} = g^e.
\]
Hence, $3^{2^{-1}} = w^{3 \frac{e+1}{2}}$, i.e. $3^{2^{-1}} = w^{e+3}$. Theorem 1 (c) implies that $e \not\equiv 3u \equiv 0 \mod 4$. Then $3^{2^{-1}} \neq w$.

whereupon
\[
3^{2^{-1}} \equiv -1 \pmod{p}.
\]

If $u \equiv 0 \mod 4$, the same above calculation implies that $3^{2^{-1}} \equiv -1 \pmod{p}$.

For the reciprocal, suppose that $q \not\equiv 1 \pmod{8}$, $gcd(u, 2^{-1}) = 1$, $u$ is even and $3^{2^{-1}} \equiv -1 \mod p$.

To prove that $h(x)$ is a permutation polynomial we have only to prove that $f(w^i) \neq 0$, for $0 \leq i < 4$ and $ind_g \frac{f(w^i)}{f(w^j)} \neq u(j - i) \mod 4$, for all $0 \leq i \leq j < 4$.

We have $f(w^i) = w^{2i} + w^i + 1$. Hence
\[
f(w^i) = \begin{cases} 3 & \text{if } i = 0 \\ w & \text{if } i = 1 \\ 1 & \text{if } i = 2 \\ w^3 & \text{if } i = 3 \end{cases}
\]

Then $f(w^i) \neq 0$, for every $0 \leq i < 4$. We have
\[
\frac{f(w^i)}{f(w^j)} = \frac{w^{2i} + w^i + 1}{w^{2j} + w^j + 1}.
\]
If $i = 0$ and $j = 1$, then
\[
\frac{f(w^i)}{f(w^j)} = \frac{3}{w} = g^e,
\]
where $e = Ind_g \left( \frac{w^i}{w^j} \right)$. Hence,
\[
3^{2^{-1}} = w^\frac{3^{-1}}{4} (g^e)^{\frac{3^{-1}}{4}},
\]
whereupon

\[ 3^{\frac{q-1}{4}} = w^{\frac{q-1}{4}} w^e. \]

Then

\[ -1 = w^{\frac{q-1}{4} + e} w^e, \quad (45) \]

which implies that \( \frac{q-1}{4} + e \equiv 2 \mod 4. \)

We have \( q \not\equiv 1 \mod 8, \) then \( \frac{q-1}{4} = 1 \) or \( 3 \mod 4. \) Hence, equation (6) implies that

\[ e \equiv 2 - \frac{q-1}{4} \equiv \pm 1 \mod 4. \]

Since \( u \) is even, then \( e \not\equiv u(j - i) \mod 4. \) For the other possibilities of \( i \) and \( j, \) the same calculations give the result.

**Concluding remarks.**

1. The study of permutation polynomials of the form \( g(x) = x^u(1 \pm x^{\frac{q-1}{4}} \pm x^{\frac{q-3}{4}}) \) is given by Theorems 2 and 3. In fact, a change of variable \( x \to -x \) in \( h(x) \) in Theorem 2 and Theorem 4 gives all the possibilities for \( g(x). \)

2. The condition \( p \neq 5 \) in Theorem 2 is necessary, in fact, if \( p = 5 \) in Theorem 2, we have \( f(w) = 0 \) or \( f(w^3) = 0, \) implying that \( h(x) \) is not a permutation polynomial of \( \mathbb{F}_q. \)

**References**


Expectation Identities from extended Burr XII Distribution based on generalized order Statistics and Characterization

Devendra Kumar\(^1\) and Anju Goyal

Department of Statistics, Amity Institute of Applied Sciences
Amity University, Noida-201 303, India

Department of Statistics, Panjab University, Chandigarh - 160014, India

Abstract

In this paper we consider extended Burr XII distribution. Exact expressions and some recurrence relations for single and product moments of generalized order statistics are derived. The relations for order statistics and upper records are deduced from the relations derived. Further, the distribution has been characterized on using conditional expectation of function of generalized order statistics and a recurrence relation for single moments of generalized order statistics; also, we use the established explicit expressions to calculate the first four moments and variances of order statistics and upper records.

Keywords: Generalized order statistics, order statistics, record values, extended Burr XII distribution, single and product moment, recurrence relations and characterization.


1 Introduction

Kamps [9] introduced the concept of generalized order statistics (gos) as follows: Let \( X_1, X_2, \ldots \) be a sequence of independent and identically distributed (iid) random variables (rv) with absolutely continuous distribution function (df) \( F(x) \) and probability density function (pdf), \( f(x), x \in (\alpha, \beta) \). Let \( n \in N, k \geq 1, m \in \mathbb{R} \), be the parameters such that

\(^1\)Corresponding author. E-mail addresses: devendrastats@gmail.com
\[ \gamma_r = k + (n - r)(m + 1) > 0, \quad \text{for all } r \in 1, 2, \ldots, n - 1, \]

where \( M_r = \sum_{j=r}^{n-1} m j. \) Then \( X(1, n, m, k), \ldots, X(n, n, m, k) \) are called \( \text{gos} \) if their joint pdf is given by

\[
k(\prod_{j=1}^{n-1} \gamma_j)(\prod_{i=1}^{n-1}[1 - F(x_i)]^m f(x_i))[1 - F(x_n)]^{k-1} f(x_n) \]  \hspace{1cm} (1.1)

on the cone \( F^{-1}(0) \leq x_1 \leq x_2 \leq \ldots \leq x_n \leq F^{-1}(1). \)

The model of \( \text{gos} \) contains as special cases, order statistics, record values, sequential order statistics.

Choosing the parameters appropriately (Cramer, [4]), we get:

Table 1.1: Variants of the generalized order statistics

<table>
<thead>
<tr>
<th>i) Sequential order statistics</th>
<th>( \gamma_n = k )</th>
<th>( \gamma_r )</th>
<th>( m_r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>ii) Ordinary order statistics</td>
<td>( \alpha_n )</td>
<td>( (n - r + 1)\alpha_r )</td>
<td>( \gamma_r - \gamma_{r+1} - 1 )</td>
</tr>
<tr>
<td>ii) Record values</td>
<td>1</td>
<td>( n - r + 1 )</td>
<td>0</td>
</tr>
<tr>
<td>iv) Progressively type II censored order statistics</td>
<td>( R_n + 1 )</td>
<td>( n - r + 1 + \sum_{j=r}^{n} R_j )</td>
<td>( R_r )</td>
</tr>
<tr>
<td>v) Pfeifer’s record values</td>
<td>( \beta_n )</td>
<td>( \beta_r )</td>
<td>( \beta_r - \beta_{r+1} - 1 )</td>
</tr>
</tbody>
</table>

For simplicity we shall assume \( m_1 = m_2 = \ldots = m_{n-1} = m. \)

The pdf of the \( r \)-th gos, \( X(r, n, m, k), 1 \leq r \leq n \) is

\[
f_X(r,n,m,k)(x) = \frac{C_{r-1}}{(r-1)!}[\hat{F}(x)]^{\gamma_{r-1}} f(x) g_m^{r-1}(F(x)) \]  \hspace{1cm} (1.2)

and the joint pdf of \( X(r, n, m, k) \) and \( X(s, n, m, k), 1 \leq r < s \leq n \) is

\[
f_{X(r,n,m,k),X(s,n,m,k)}(x, y) = \frac{C_{s-1}}{(s-1)!(s-r-1)!}[\hat{F}(x)]^m f(x) g_m^{s-r-1}(F(x))
\times[h_m(F(y)) - h_m(F(x))]^{s-r-1}[\hat{F}(y)]^{\gamma_{s-1}} f(y), \quad x > y, \]  \hspace{1cm} (1.3)

where

\[
\hat{F}(x) = 1 - F(x), \quad C_{r-1} = \prod_{i=1}^{r} \gamma_i, \quad \gamma_i = k + (n-i)(m+1),
\]
Devendra Kumar, Anju Goyali

\[
h_m(x) = \begin{cases} 
\frac{1}{m+1} (1-x)^{m+1}, & m \neq -1 \\
-m(1-x), & m = -1 
\end{cases}
\]

and

\[
g_m(x) = h_m(x) - h_m(0), \quad x \in [0, 1).
\]

Recurrence relations are interesting in their own right. They are useful in reducing the number of operations necessary to obtain a general form for the function under consideration. Furthermore, they are used in characterizing the distributions, which in important area, permitting the identification of population distribution from the properties of the sample. Recurrence relations and identities have attained importance reduces the amount of direct computation and hence reduces the time and labour. They express the higher order moments in terms of order moments and hence make the evaluation of higher order moments easy and provide some simple checks to test the accuracy of computation of moments of order statistics.

Kamps [9] investigated recurrence relations for moments of generalized order statistics based on non-identically distributed random variables, which contains order statistics and record values as special cases. Cramer and Kamps [5] derived relations for expectations of functions of \(gos\) within a class of distributions including a variety of identities for single and product moments of ordinary order statistics and record values as particular cases. Various developments on \(gos\) and related topics have been studied by Kamps and Gather [8], Keseling [11], Pawlas and Szynal [18], Ahmed and Fawzy [1], Ahmed [2], Khan, et al. [12], Kumar [13], [14], [15] among others. Kamps [10] investigated the importance of recurrence relations of order statistics in characterization.

The aim of the present study is to give some explicit expressions and recurrence relations for single and product moments of \(gos\) from extended Burr XII distribution. In section 2, we give an explicit expressions and recurrence relations for single moments of extended Burr XII distribution. Then we show that the results for order statistics and the record values are deduced as in special cases. In section 3, we give a explicit expressions and recurrence relations for product moments of extended Burr XII distribution and we show that the results for order statistics and the record values are deduced as in special cases. In last section of the paper, we prove a two characterization results of extended Burr XII distribution based on conditional expectation of function of \(gos\) and a recurrence relation for single moments of \(gos\), also, we use the established explicit expressions to calculate the first four moments and variances of order statistics and upper records for \(n = 1(1)5\) (see Table 2.1, 2.2, 2.3 and 2.4).

A random variable \(X\) is said to have extended Burr XII distribution (Madulkar et al., [17]) if its pdf is of the form
\[ f(x) = \frac{\alpha}{\beta} \left( \frac{x}{\beta} \right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^\alpha}, \quad x > 0, \quad \alpha, \beta > 0 \] (1.4)

and the corresponding \( df \) is

\[ F(x) = 1 - e^{-\left(\frac{x}{\beta}\right)^\alpha}, \quad x > 0, \quad \alpha, \beta > 0. \] (1.5)

It is easy to see that

\[ f(x) = \frac{\alpha}{\beta} \left( \frac{x}{\beta} \right)^{\alpha-1} \bar{F}(x). \] (1.6)

The Weibull and exponential distributions are considered as special cases of the extended Burr XII distribution when \( \beta = 1 \) and \( \alpha = 1, \beta = 1 \), respectively. The extended Burr XII distribution is promoted by applying a power transformation to the generalized Pareto distribution, which is a theoretical distribution to model the values over thresholds. Therefore, the extended Burr XII distribution may be used as a unified distribution to model both block maxima (e.g. annual data) and exceedences over threshold (e.g. daily data) and can be applied to data with moderate sampling frequencies (e.g. monthly data) where seasonality may appear.

2 Relations for single moments

First we prove some results, which may be needed subsequently.

**Lemma 2.1:** For the extended Burr XII distribution as given in (1.5) and any non-negative finite integers \( a \) and \( b \) with \( m \neq -1 \)

\[ I_j(a, 0) = \frac{\beta^j \Gamma(1 + j/\alpha)}{(a + 1)^{1+j/\alpha}}, \] (2.1)

where

\[ I_j(a, b) = \int_0^\infty x^j [\bar{F}(x)]^a f(x) g_m(F(x)) dx. \] (2.2)

**Proof:** From (2.2)

\[ I_j(a, 0) = \int_0^\infty x^j [\bar{F}(x)]^a f(x) dx. \] (2.3)

Making the substitution \( t = -ln\bar{F}(x) \) in (2.3) and simplification the resulting expression we get the result given in (2.1).

**Lemma 2.2:** For the extended Burr XII distribution as given in (1.5) and any non-negative finite integers \( a \) and \( b \)

\[ I_j(a, b) = \frac{1}{(m + 1)^b} \sum_{u=0}^b (-1)^u \binom{b}{u} I_j(a + u(m + 1), 0) \] (2.4)
\[ \beta^j \frac{1}{(m+1)^b} \sum_{u=0}^{b} (-1)^u \binom{b}{u} \frac{\Gamma(1 + j/\alpha)}{[a + u(m+1) + 1]^{1+j/\alpha}}, \quad m \neq -1 \]  

(2.4)

where \( I_j(a,b) \) is as given in (2.2).

**Proof:** On expanding \( g_m^b(F(x)) = \left[ \frac{1}{m+1} \{1 - (\bar{F}(x))^{m+1}\} \right]^b \) binomially in (2.2), we get when \( m \neq -1 \)

\[ I_j(a,b) = \frac{1}{(m+1)^b} \sum_{u=0}^{b} (-1)^u \binom{b}{u} \int_0^\infty x^j[\bar{F}(x)]^{a+u(m+1)}dx \]

\[ = \frac{1}{(m+1)^b} \sum_{u=0}^{b} (-1)^u \binom{b}{u} I_j(a + u(m+1), 0). \]

Making use of Lemma 2.1, we derive the result given in (2.5).

**Theorem 2.1:** For the extended Burr XII distribution as given in (1.5) and \( 1 \leq r \leq n \), \( k = 1, 2, \ldots \) and \( m \neq -1 \)

\[ E[X_j^j(r, n, m, k)] = \frac{\beta^j C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} \frac{\Gamma(1 + j/\alpha)}{(\gamma_{r-u})^{1+j/\alpha}}, \]

\[ \alpha > j, \quad j = 0, 1, 2, \ldots \]  

(2.6)

**Proof:** From (1.2), we have

\[ E[X_j^j(r, n, m, k)] = \frac{C_{r-1}}{(r-1)!} \int_0^\infty x^j[\bar{F}(x)]^{\gamma_{r-1}} f(x)g_m^{r-1}(F(x))dx. \]

\[ = \frac{C_{r-1}}{(r-1)!} I_j(\gamma_{r-1}, r-1). \]

Making use of Lemma 2.2, we establish the result given in (2.6).

and when \( m = -1 \) that

\[ E[X_j^j(r, n, m, k)] = \frac{0}{0} \quad \text{as} \quad \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} = 0 \]

Since (2.7) is of the form \( \frac{0}{0} \) at \( m = -1 \), therefore, we have

\[ E[X_j^j(r, n, m, k)] = A \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} \frac{[k + (n - r + u)(m+1)]^{-(1+j/\alpha)}}{(m+1)^{r-1}}, \]

\[ A = \frac{\beta^j C_{r-1}}{(r-1)!} \Gamma(1 + j/\alpha). \]  

(2.7)
Differentiating numerator and denominator of (2.7) \((r - 1)\) times with respect to \(m\), we get

\[
E[X^j(r, n, m, k)] = A \sum_{u=0}^{r-1} (-1)^{u+(r-1)} \binom{r-1}{u} \frac{(1 + j/\alpha)(2 + j/\alpha) \ldots (r - 1 + j/\alpha)(n - r + u)r^{-1}}{(r - 1)! [k + (n - r + u)(m + 1)]^{r+j/\alpha}}.
\]

On applying L’ Hospital rule, we have

\[
\lim_{m \to -1} E[X^j(r, n, m, k)] = A \frac{(1 + j/\alpha)(2 + j/\alpha) \ldots (r - 1 + j/\alpha)}{(r - 1)! [k + (n - r + u)(m + 1)]^{r+j/\alpha}} \sum_{u=0}^{r-1} (-1)^{u} \binom{r-1}{u} (r - n - u)^{r-1}.
\]

But for all integers \(n \geq 0\) and for all real numbers \(x\), we have Ruiz [19]

\[
\sum_{i=0}^{n} (-1)^{i} \binom{n}{i} (x - i)^{n} = n!
\]

Now substituting (2.9) in (2.8) and simplifying, we find that

\[
E[X^j(r, n, -1, k)] = E[(Z_r^{(k)})^j] = \frac{\beta^j \Gamma(r + j/\alpha)}{k^{j/\alpha} (r - 1)!^r},
\]

as obtained by Kamps [9] and Grudzien and Syznal [6] for exponential distribution at \(\alpha = 1, \beta = 1/c\) and \(\alpha = 1, j = 1\) and \(\beta = 1/c\) respectively.

**Special cases**

i) Putting \(m = 0, k = 1\) in (2.6), the exact expression for the single moments of order statistics of the extended Burr XII distribution can be obtained as

\[
E(X^j_{r:n}) = \beta^j C_{r:n} \sum_{u=0}^{r-1} (-1)^{u} \binom{r-1}{u} \frac{\Gamma(1 + j/\alpha)}{(n - r + u + 1)^{1+j/\alpha}}.
\]

where

\[
C_{r:n} = \frac{n!}{(r - 1)!(n - r)!}.
\]

Which is the result obtained by Lieblein [16] for exponential distribution at \(\alpha = 1\) and \(\beta = 1/c\).
ii) Setting $k = 1$ in (2.10), we get the exact expression for the single moments of records for the extended Burr XII distribution can be obtained as

$$E[X^j(r, n, -1, 1)] = E[(Y_r)^j] = \frac{\beta \Gamma(r + j/\alpha)}{(r - 1)!},$$

as obtained by Ahsanullah [3] for exponential distribution at $j = 1$, $\alpha = 1$ and $\beta = 1/c$.

Recurrence relations for single moments of $gos$ from (1.2) can be obtained in the following theorem.

**Theorem 2.2:** For the distribution given in (1.5) and for $2 \leq r \leq n$, $n \geq 2$ and $k = 1, 2, \ldots$

$$E[X^{j+1}(r, n, m, k)] = \frac{\beta^\alpha(j + 1)}{\alpha \gamma_r} \left\{ E[X^{j-\alpha+1}(r, n, m, k)] + E[X^{j+1}(r-1, n, m, k)] \right\}.$$  \hspace{1cm} (2.11)

**Proof:** From (1.2) and (1.6), we have

$$E[X^{j-\alpha+1}(r, n, m, k)] = \frac{\alpha C_{r-1}}{\beta^\alpha (r - 1)!} \int_0^\infty x^j [\bar{F}(x)]^{\gamma - 1} g_m^{-1}(F(x)) dx.$$

Integrating by parts treating $x^j$ for integration and rest of the integrand for differentiation, we get

$$E[X^{j-\alpha+1}(r, n, m, k)] = \frac{\alpha \gamma_r}{\beta^\alpha (j + 1)} \left\{ \frac{C_{r-2}}{(r - 2)!} \int_0^\infty x^{j+1} [\bar{F}(x)]^{\gamma - 1} f(x) dx \right. \times g_m^{r-2}(F(x)) dx + \frac{C_{r-1}}{(r - 1)!} \int_0^\infty x^{j+1} [\bar{F}(x)]^{\gamma - 1} f(x) g_m^{-1}(F(x)) dx \right\}$$

and hence the result.

**Remark 2.1:** Putting $j = 0$, $\beta = 1/c$ and $\alpha = 1$ in (2.11), the result for single moments of $gos$, obtained by Kamps [9] for exponential distribution is deduced. **Remark 2.2:** Putting $m = 0$, $k = 1$, in (2.11), we obtain a recurrence relation for single moments of order statistics of the extended Burr XII distribution in the form

$$E(X_{r,n}^{j+1}) = \frac{\beta^\alpha(j + 1)}{\alpha (n - r + 1)} \left\{ E(X_{r,n}^{j-\alpha+1}) + E(X_{r-1,n}^{j+1}) \right\}.$$

**Remark 2.3:** Setting $m = -1$ in Theorem 2.2, we get a recurrence relation for single moments of $k$ record values from extended Burr XII distribution in the form

$$E[X^{j+1}(r, n, -1, k)] = \frac{\beta^\alpha(j + 1)}{\alpha k} \left\{ E[X^{j-\alpha+1}(r, n, -1, k)] + E[X^{j+1}(r-1, n, -1, k)] \right\}.$$
as obtained by Kamps [9] for exponential distribution at $j + 1 = j$, $\beta = 1/c$ and $\alpha = 1$.

Table 2.1 First four moments of order statistics

<table>
<thead>
<tr>
<th>$n$</th>
<th>$r$</th>
<th>$j = 1, \beta = 2$</th>
<th>$j = 1, \beta = 3$</th>
<th>$j = 1, \beta = 4$</th>
<th>$j = 1, \beta = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$\alpha = 1$</td>
<td>$\alpha = 2$</td>
<td>$\alpha = 1$</td>
<td>$\alpha = 2$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2.00000</td>
<td>1.77245</td>
<td>3.00000</td>
<td>2.65868</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1.00000</td>
<td>1.25331</td>
<td>1.50000</td>
<td>1.87997</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>3.00000</td>
<td>2.29159</td>
<td>4.50000</td>
<td>3.43739</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0.66667</td>
<td>1.02333</td>
<td>1.00000</td>
<td>1.53499</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1.66667</td>
<td>1.71329</td>
<td>2.50000</td>
<td>2.56993</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>3.66667</td>
<td>2.58075</td>
<td>5.50000</td>
<td>3.87112</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>0.50000</td>
<td>0.88623</td>
<td>0.75000</td>
<td>1.32934</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1.16667</td>
<td>1.42463</td>
<td>1.75000</td>
<td>2.15194</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>2.16667</td>
<td>1.99195</td>
<td>3.25000</td>
<td>2.98793</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>4.16667</td>
<td>2.77701</td>
<td>6.25000</td>
<td>4.16552</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>0.40000</td>
<td>0.79267</td>
<td>0.60000</td>
<td>1.18900</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.90000</td>
<td>1.26047</td>
<td>1.35000</td>
<td>1.89071</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>1.56667</td>
<td>1.69586</td>
<td>2.35000</td>
<td>2.54378</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>2.56667</td>
<td>2.18935</td>
<td>3.85000</td>
<td>3.28402</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>4.56667</td>
<td>2.92393</td>
<td>6.85000</td>
<td>4.38589</td>
</tr>
</tbody>
</table>

|     |     | $\alpha = 1$     | $\alpha = 2$     | $\alpha = 1$     | $\alpha = 2$     |
| 1   | 1   | 4.00000          | 3.54491           | 5.00000          | 4.43113          |
| 2   | 1   | 2.00000          | 2.50663           | 2.50000          | 3.13329          |
|     | 2   | 6.00000          | 4.58319           | 7.50000          | 5.72898          |
| 3   | 1   | 1.33333          | 2.04665           | 1.66667          | 2.55832          |
|     | 2   | 3.33333          | 1.71329           | 2.50000          | 2.56993          |
|     | 3   | 3.66667          | 3.42658           | 4.16667          | 4.28322          |
| 4   | 1   | 1.00000          | 1.77245           | 1.25000          | 2.21557          |
|     | 2   | 2.33333          | 2.86925           | 2.91667          | 3.58657          |
|     | 3   | 4.33333          | 3.98390           | 5.41667          | 4.97988          |
|     | 4   | 8.33333          | 5.55402           | 10.41667         | 6.94253          |
| 5   | 1   | 0.80000          | 1.58533           | 1.00000          | 1.98166          |
|     | 2   | 1.80000          | 2.52095           | 2.25000          | 3.15118          |
|     | 3   | 3.13333          | 3.39171           | 3.91667          | 4.23904          |
|     | 4   | 5.13333          | 4.37870           | 6.41667          | 5.47337          |
|     | 5   | 9.13333          | 5.84785           | 11.41667         | 7.30981          |

continue...
\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
n & r & j = 2, \beta = 3 & & j = 2, \beta = 4 & \\
& & \alpha = 1 & \alpha = 2 & \alpha = 1 & \alpha = 2 \\
\hline
1 & 1 & 18.00000 & 9.00000 & 32.00000 & 16.00000 \\
2 & 1 & 45.00000 & 4.50000 & 8.00000 & 8.00000 \\
& 2 & 31.50000 & 13.50000 & 56.00000 & 24.00000 \\
3 & 1 & 2.00000 & 3.00000 & 3.55556 & 5.33333 \\
& 3 & 42.50000 & 16.50000 & 75.55556 & 29.33333 \\
4 & 1 & 1.12500 & 2.25000 & 2.00000 & 4.00000 \\
& 2 & 4.62500 & 5.25000 & 8.22222 & 9.33333 \\
& 4 & 51.87500 & 18.75000 & 92.22222 & 33.33333 \\
5 & 1 & 0.72000 & 1.80000 & 1.28000 & 3.20000 \\
& 2 & 2.74500 & 4.05000 & 4.88000 & 7.20000 \\
& 3 & 7.44500 & 7.05000 & 13.23556 & 12.53333 \\
& 4 & 18.99500 & 11.55000 & 33.76889 & 20.53333 \\
& 5 & 60.09500 & 20.55000 & 106.83556 & 36.53333 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
n & r & j = 2, \beta = 5 & & j = 3, \beta = 4 & \\
& & \alpha = 1 & \alpha = 2 & \alpha = 1 & \alpha = 2 \\
\hline
1 & 1 & 50.00000 & 25.00000 & 384.00000 & 85.07778 \\
2 & 1 & 12.50000 & 12.50000 & 48.00000 & 30.07954 \\
& 2 & 87.50000 & 37.5000056 & 720.00000 & 140.07600 \\
3 & 1 & 5.55556 & 8.33333 & 16.37323 & 16.37323 \\
& 2 & 26.38889 & 20.83333 & 115.55556 & 57.49216 \\
& 3 & 118.05556 & 45.83333 & 1022.22200 & 181.36800 \\
4 & 1 & 3.12500 & 6.25000 & 6.00000 & 10.63472 \\
& 2 & 12.84722 & 14.58333 & 38.88889 & 33.58874 \\
& 3 & 39.93056 & 27.08333 & 192.22222 & 81.39559 \\
& 4 & 144.09722 & 52.08333 & 1298.88890 & 214.69210 \\
5 & 1 & 2.00000 & 5.00000 & 3.07200 & 7.60959 \\
& 2 & 7.62500 & 11.25000 & 17.71200 & 22.73526 \\
& 3 & 20.68056 & 19.58333 & 70.65422 & 49.86896 \\
& 4 & 39.93056 & 27.08333 & 192.22222 & 81.39559 \\
& 5 & 166.93056 & 57.08333 & 1298.88890 & 214.69210 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
n & r & j = 3, \beta = 5 & & j = 4, \beta = 5 & \\
& & \alpha = 1 & \alpha = 2 & \alpha = 1 & \alpha = 2 \\
\hline
1 & 1 & 750.00000 & 166.16760 & 1500.00000 & 1250.00000 \\
2 & 1 & 93.75000 & 58.74910 & 937.50000 & 312.50000 \\
& 2 & 1406.25000 & 273.58600 & 29062.50000 & 2187.50000 \\
3 & 1 & 27.77778 & 31.97896 & 185.18520 & 138.88890 \\
& 2 & 225.69440 & 112.28940 & 2442.12960 & 659.72220 \\
& 3 & 1996.52780 & 354.23430 & 4237.68500 & 181.36800 \\
& 4 & 2536.89240 & 419.32050 & 55057.14700 & 3602.43100 \\
\hline
\end{array}
\]
**Table 2.2** Variance of order statistics

<table>
<thead>
<tr>
<th>n</th>
<th>r</th>
<th>( \beta = 3 )</th>
<th>( \beta = 4 )</th>
<th>( \beta = 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>( \alpha = 1 )</td>
<td>( \alpha = 2 )</td>
<td>( \alpha = 1 )</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>9.00000</td>
<td>1.93142</td>
<td>16.00000</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>2.25000</td>
<td>0.96571</td>
<td>4.00000</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>11.25000</td>
<td>1.68435</td>
<td>20.00000</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1.00000</td>
<td>0.64381</td>
<td>1.77779</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>3.25000</td>
<td>0.89546</td>
<td>5.77780</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>12.25000</td>
<td>1.51443</td>
<td>21.77783</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0.56250</td>
<td>0.48286</td>
<td>1.00000</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1.56250</td>
<td>0.61915</td>
<td>2.77779</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>3.81250</td>
<td>0.82227</td>
<td>6.77781</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>12.81250</td>
<td>1.39844</td>
<td>22.77783</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>0.36000</td>
<td>0.38628</td>
<td>0.64000</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.92250</td>
<td>0.47522</td>
<td>1.64000</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>1.92250</td>
<td>0.57918</td>
<td>3.41780</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>4.17250</td>
<td>0.76521</td>
<td>7.41781</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>13.17250</td>
<td>1.31397</td>
<td>23.41784</td>
</tr>
</tbody>
</table>

**Table 2.3** First four moments of upper records

<table>
<thead>
<tr>
<th>j = 1, ( \beta = 2 )</th>
<th>j = 1, ( \beta = 3 )</th>
<th>j = 1, ( \beta = 4 )</th>
<th>j = 1, ( \beta = 5 )</th>
<th>j = 2, ( \beta = 3 )</th>
<th>j = 2, ( \beta = 4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha = 1 )</td>
<td>( \alpha = 2 )</td>
<td>( \alpha = 1 )</td>
<td>( \alpha = 2 )</td>
<td>( \alpha = 1 )</td>
<td>( \alpha = 2 )</td>
</tr>
<tr>
<td>1</td>
<td>2.00000</td>
<td>1.77245</td>
<td>3.00000</td>
<td>2.65868</td>
<td>5.00000</td>
</tr>
<tr>
<td>2</td>
<td>4.00000</td>
<td>2.65868</td>
<td>6.00000</td>
<td>3.98802</td>
<td>12.00000</td>
</tr>
<tr>
<td>4</td>
<td>8.00000</td>
<td>3.87724</td>
<td>12.00000</td>
<td>5.81586</td>
<td>30.00000</td>
</tr>
<tr>
<td>5</td>
<td>10.00000</td>
<td>4.36190</td>
<td>15.00000</td>
<td>6.54285</td>
<td>50.00000</td>
</tr>
<tr>
<td>( \alpha = 1 )</td>
<td>( \alpha = 2 )</td>
<td>( \alpha = 1 )</td>
<td>( \alpha = 2 )</td>
<td>( \alpha = 1 )</td>
<td>( \alpha = 2 )</td>
</tr>
<tr>
<td>1</td>
<td>4.00000</td>
<td>3.54491</td>
<td>5.00000</td>
<td>4.43113</td>
<td>16.00000</td>
</tr>
<tr>
<td>2</td>
<td>8.00000</td>
<td>5.31736</td>
<td>10.00000</td>
<td>6.64670</td>
<td>32.00000</td>
</tr>
<tr>
<td>3</td>
<td>12.00000</td>
<td>6.64670</td>
<td>15.00000</td>
<td>8.30838</td>
<td>64.00000</td>
</tr>
<tr>
<td>4</td>
<td>16.00000</td>
<td>7.75449</td>
<td>20.00000</td>
<td>5.81586</td>
<td>128.00000</td>
</tr>
<tr>
<td>5</td>
<td>20.00000</td>
<td>8.72380</td>
<td>25.00000</td>
<td>10.90475</td>
<td>256.00000</td>
</tr>
</tbody>
</table>

| \( \alpha = 1 \)      | \( \alpha = 2 \)      | \( \alpha = 1 \)      | \( \alpha = 2 \)      | \( \alpha = 1 \)      | \( \alpha = 2 \)      |
| 1                      | 18.00000               | 9.00000                | 32.00000               | 16.00000               | 64.00000               | 64.00000               |
| 2                      | 54.00000               | 18.00000               | 96.00000               | 32.00000               | 256.00000              | 256.00000              |
| 3                      | 108.00000              | 27.00000               | 192.00000              | 48.00000               | 512.00000              | 512.00000              |
| 4                      | 180.00000              | 36.00000               | 320.00000              | 64.00000               | 1024.00000             | 1024.00000             |
| 5                      | 270.00000              | 45.00000               | 480.00000              | 80.00000               | 2048.00000             | 2048.00000             |
continue...

<table>
<thead>
<tr>
<th>r</th>
<th>j = 2, β = 5</th>
<th>j = 3, β = 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>α = 1 50.00000</td>
<td>α = 1 384.00000</td>
</tr>
<tr>
<td></td>
<td>α = 2 25.00000</td>
<td>α = 2 85.07778</td>
</tr>
<tr>
<td>2</td>
<td>150.00000 50.00000</td>
<td>1536.00000 212.69450</td>
</tr>
<tr>
<td>3</td>
<td>300.00000 75.00000</td>
<td>3840.00000 372.21530</td>
</tr>
<tr>
<td>4</td>
<td>500.00000 100.00000</td>
<td>7680.00000 558.32300</td>
</tr>
<tr>
<td>5</td>
<td>750.00000 125.00000</td>
<td>13440.00000 767.69410</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>r</th>
<th>j = 3, β = 5</th>
<th>j = 4, β = 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>α = 1 750.00000</td>
<td>α = 1 15000.00000</td>
</tr>
<tr>
<td></td>
<td>α = 2 166.16760</td>
<td>α = 2 1250.00000</td>
</tr>
<tr>
<td>2</td>
<td>3000.00000 415.41890</td>
<td>75000.00000 3750.00000</td>
</tr>
<tr>
<td>3</td>
<td>7500.00000 726.98300</td>
<td>225000.000</td>
</tr>
<tr>
<td>4</td>
<td>15000.00000 1090.47500</td>
<td>525000.00000 12500.00000</td>
</tr>
<tr>
<td>5</td>
<td>26250.00000 1499.40200</td>
<td>1050000.000</td>
</tr>
</tbody>
</table>

Table 2.4 Variance of upper records

<table>
<thead>
<tr>
<th>r</th>
<th>β = 3 α = 1</th>
<th>β = 3 α = 2</th>
<th>β = 4 α = 1</th>
<th>β = 4 α = 2</th>
<th>β = 5 α = 1</th>
<th>β = 5 α = 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9.000000 1.93142</td>
<td>16.000000 3.43361</td>
<td>25.000000 5.36509</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>18.000000 2.09570</td>
<td>32.000000 3.72568</td>
<td>50.000000 5.82138</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>27.000000 2.14948</td>
<td>48.000000 3.82138</td>
<td>75.000000 5.97082</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>36.000000 2.17577</td>
<td>64.000000 3.86788</td>
<td>100.000000 6.04362</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>45.000000 2.19111</td>
<td>80.000000 3.89531</td>
<td>125.000000 6.08643</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3 Relations for product moments

Theorem 3.1: For the given extended Burr XII distribution in (1.5) and for $1 \leq r < r + 1 \leq n$, $n \geq 2$ and $m \in \mathbb{R}$

\[
E[X^i(r, n, m, k)X^{j+1}(r + 1, n, m, k)] = \frac{\beta^\alpha(j + 1)}{\gamma_{r+1}}
\]

\[
\times \left\{ E[X^{i+j+1}(r, n, m, k)] + E[X^i(r, n, m, k)X^{j+1}(r + 1, n, m, k)] \right\}
\]

(3.1)

and for $1 \leq r < s \leq n$, $s - r \geq 2$ and $i, j \geq 0$,

\[
E[X^i(r, n, m, k)X^{j+1}(s, n, m, k)] = \frac{\beta^\alpha(j + 1)}{\gamma_{s}}
\]
\[
\times \left\{ E[X^i(r, n, m, k)X^{j-\alpha+1}(r, n, m, k)] + E[X^i(r, n, m, k)X^{j+1}(s, n, m, k)] \right\}
\]

(3.2)

**Proof:** From (1.3), we have

\[
E[X^i(r, n, m, k)X^{j-\alpha+1}(s, n, m, k)] = \frac{C_s-1}{(r-1)!(s-r-1)!} \int_0^\infty x^i[F(x)]^m f(x)g_m^{-1}(F(x))I(x)dx,
\]

(3.3)

where

\[
I(x) = \int_x^\infty y^{j-\alpha+1}[h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1}f(y)dy
\]

(3.4)

upon using the relation in (1.6). Integrating now by parts treating \(y^j\) for integration and the rest of the integrand for differentiation, we obtain when \(s = r + 1\) that

\[
I(x) = \frac{\alpha\gamma_{r+1}}{\beta\alpha(j+1)} \left\{ -\frac{x^{j+1}}{\gamma_{r+1}}[\bar{F}(x)]^{\gamma_{r+1}} + \int_x^\infty y^{j+1}[\bar{F}(x)]^{\gamma_{r+1}-1}f(y)dy \right\}
\]

and when \(s > r + 1\) that

\[
I(x) = \frac{\alpha\gamma_s}{\beta\alpha(j+1)} \left\{ -\frac{(s-r-1)}{\gamma_s} \int_x^\infty y^{j+1}[h_m(F(y)) - h_m(F(x))]^{s-r-2}[\bar{F}(y)]^{\gamma_s+m} \right. \\
\left. \times f(y)dy + \int_x^\infty y^{j+1}[h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1}f(y)dy \right\}.
\]

Upon substituting the above expressions for \(I(x)\) in (3.3), we have, after simplifications, the recurrence relations (3.1) and (3.2).

**Remark 3.1:** Putting \(m = 0, k = 1\) in (3.1) and (3.2), we obtain recurrence relations for product moments of order statistics for the extended Burr XII distribution in the form

\[
E[X^i(r, n, m, k)X^{j+1}(s, n, m, k)] = \frac{\beta\alpha(j+1)}{\alpha(n-r)} E[X^i_{r,n}X^{j-\alpha+1}_{r+1,n}] + E[X^i_{r,n}X^{j+1}_{r+1,n}]
\]

and

\[
E[X^i_{r,n}X^{j+1}_{s,n}] = \frac{\beta\alpha(j+1)}{\alpha(n-s+1)} \left\{ E[X^i_{r,n}X^{j-\alpha+1}_{s,n}] + E[X^i_{r,n}X^{j+1}_{s-1,n}] \right\}.
\]
Remark 3.2: Setting $m = -1$ and $k \geq 1$ in Theorem 3.1, we get the recurrence relations for product moments of $k$ record values from extended Burr XII distribution in the form

$$E[(Z_r^{(k)})^i(Z_s^{(k)})^{j+1}] = \frac{\beta^a(j + 1)}{\alpha k} \left\{ E[(Z_r^{(k)})^i(Z_s^{(k)})^{j-a+1}] + E[(Z_r^{(k)})^i(Z_{s-1}^{(k)})^{j+1}] \right\}. $$

Remark 3.3: At $i = 0$ in (3.2), recurrence relations for product moments reduces to relations for single moments as obtained in (2.11).

3 Characterization

Let $X(r,n,m,k), r = 1, 2, \ldots, n$ be gos from a continuous population with df $F(x)$ and pdf $f(x)$, then the conditional pdf of $X(s,n,m,k)$ given $X(r,n,m,k) = x$, $1 \leq r < s \leq n$, in view of (1.2) and (1.3), is

$$f_{X(s,n,m,k)|X(r,n,m,k)}(y|x) = \frac{C_{s-1}}{(s-r-1)!C_{r-1}} \times \frac{[h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{-1}}{[F(x)]^x f(y), \ x < y.} \quad (4.1)$$

Theorem 4.1: Let $X$ be a non-negative random variable having an absolutely continuous distribution function $F(x)$ with $F(0) = 0$ and $0 < F(x) < 1$ for all $x > 0$, then

$$E[\xi(X(r,n,m,k))|X(l,n,m,k) = x] = e^{-(x/\beta)^a} \prod_{j=1}^{s-l} \left( \frac{\gamma_{l+j}}{\gamma_{l+j}+1} \right), \ l = r, r+1 $$

if and only if

$$\bar{F}(x) = e^{-(x/\beta)^a}, \ x > 0, \ \alpha, \beta > 0. \quad (4.2)$$

where

$$\xi(x) = e^{-(y/\beta)^a}$$

Proof: From (4.1), we have

$$E[\xi(X(r,n,m,k))|X(l,n,m,k) = x] = \frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}} \int_x^\infty e^{-\frac{y}{2\beta}} \left[ \left( \frac{F(y)}{F(x)} \right)^{\frac{m+1}{\gamma-1}} \left( \frac{F(y)}{F(x)} \right)^{\gamma_{l+j} - \frac{1}{\gamma_{l+j}} f(x)} \right] dy. \quad (4.3)$$

By setting $u = \frac{F(y)}{F(x)} = e^{-(y/\beta)^a}$ from (1.4) in (4.3), we obtain

$$E[\xi(X(r,n,m,k))|X(r,n,m,k) = x] = \frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}} \int_x^\infty e^{-\frac{y}{2\beta}} \left[ \left( \frac{F(y)}{F(x)} \right)^{\frac{m+1}{\gamma-1}} \left( \frac{F(y)}{F(x)} \right)^{\gamma_{l+j} - \frac{1}{\gamma_{l+j}} f(x)} \right] dy. \quad (4.3)$$
\[
\times e^{-\frac{x}{\beta}\alpha} \int_0^1 u^{\gamma s} (1 - u^{m+1})^{s-r-1} du.
\]

(4.4)

Again by setting \( t = u^{m+1} \) in (4.4), we get

\[
E[\xi(X(r, n, m, k))|X(r, n, m, k) = x] = \frac{C_{s-1}}{(s - r - 1)!C_{r-1}(m + 1)^{s-r}}
\times e^{-\frac{x}{\beta}\alpha} \int_0^1 t^{k+1+n-s-1} (1 - t)^{s-r-1} dt
\]

\[
= e^{-\frac{x}{\beta}\alpha} \prod_{j=1}^{s-r} \left( \frac{\gamma r + j}{\gamma r + j + 1} \right)
\]

and hence the relation in (4.2).

To prove sufficient part, we have from (4.1) and (4.2)

\[
\frac{C_{s-1}}{(s - r - 1)!C_{r-1}(m + 1)^{s-r-1}} \int_x^{\infty} e^{-\frac{y}{\beta}\alpha} [\bar{F}(x)]^{m+1} - (\bar{F}(y))^{m+1}]^{s-r-1}
\times [\bar{F}(y)]^{\gamma r} f(y) dy = [\bar{F}(x)]^{\gamma r+1} H_r(x),
\]

(4.5)

where

\[
H_r(x) = e^{-\frac{x}{\beta}\alpha} \prod_{j=1}^{s-r} \left( \frac{\gamma r + j}{\gamma r + j + 1} \right).
\]

Differentiating (4.5) both the sides with respect to \( x \), we get

\[
\frac{C_{s-1}[\bar{F}(x)]^m f(x)}{(s - r - 2)!C_{r-1}(m + 1)^{s-r-2}} \int_x^{\infty} e^{-\frac{y}{\beta}\alpha} [\bar{F}(x)]^{m+1} - (\bar{F}(y))^{m+1}]^{s-r-2}
\times [\bar{F}(y)]^{\gamma r-1} f(y) dy = H_{r+1}(x)[\bar{F}(x)]^{\gamma r+1} - \gamma_{r+1} H_r(x)[\bar{F}(x)]^{\gamma r+1-1} f(x)
\]

or

\[
\gamma_{r+1} H_{r+1}(x)[\bar{F}(x)]^{\gamma r+2+m} f(x)
\]

\[
= H'_r(x)[\bar{F}(x)]^{\gamma r+1} - \gamma_{r+1} H_r(x)[\bar{F}(x)]^{\gamma r+1-1} f(x).
\]

Therefore

\[
\frac{f(x)}{\bar{F}(x)} = - \frac{H'_r(x)}{\gamma_{r+1}[H_{r+1}(x) - H_r(x)]} = \frac{\alpha}{\beta} \left( \frac{x}{\beta} \right)^{\alpha-1}
\]

which proves that

\[
\bar{F}(x) = e^{-\frac{x}{\beta}\alpha}, \quad x > 0, \quad \alpha, \beta > 0.
\]
Theorem 4.2: Let $X$ be a non-negative random variable having an absolutely continuous distribution function $F(x)$ with $F(0) = 0$ and $0 < F(x) < 1$ for all $x > 0$, then

$$E[X^{j+1}(r, n, m, k)] = \frac{\beta^\alpha(j + 1)}{\alpha \gamma_r}\left\{E[X^{j-\alpha+1}(r, n, m, k)] + E[X^{j+1}(r-1, n, m, k)]\right\}. \tag{4.6}$$

if and only if

$$\bar{F}(x) = e^{-(x/\beta)\alpha}, \quad x > 0, \quad \alpha, \beta > 0.$$

Proof: The necessary part follows immediately from equation (2.11). On the other hand if the recurrence relation in equation (4.6) is satisfied, then on using equations (1.2), we have

$$\frac{C_{r-1}}{(r-1)!} \int_0^{\infty} x^{j+1}[\bar{F}(x)]^{\gamma_r} f(x) g_m^{r-1}(F(x))dx$$

$$= \frac{\beta^\alpha(r - 1)(j + 1)C_{r-1}}{\alpha \gamma_r(r - 1)!} \int_0^{\infty} x^{j+1}[\bar{F}(x)]^{\gamma_r+m} f(x) g_m^{r-2}(F(x))dx$$

$$+ \frac{\beta^\alpha(j + 1)C_{r-1}}{\alpha \gamma_r(r - 1)!} \int_0^{\infty} x^{j-\alpha+1}[\bar{F}(x)]^{\gamma_r} f(x) g_m^{r-1}(F(x))dx$$

$$= \frac{\beta^\alpha(j + 1)C_{r-1}}{\alpha \gamma_r(r - 1)!} \int_0^{\infty} x^{j+1}[\bar{F}(x)]^{\gamma_r} f(x) g_m^{r-2}(F(x))dx$$

$$\times \left\{(r - 1)[\bar{F}(x)]^m - \frac{\gamma_r g_m(F(x))}{\bar{F}(x)}\right\}dx. \tag{4.7}$$

Let

$$h(x) = [\bar{F}(x)]^{\gamma_r} g_m^{r-1}(F(x)) \tag{4.8}$$

$$h'(x) = [\bar{F}(x)]^{\gamma_r} f(x) g_m^{r-2}(F(x))\left\{(r - 1)[\bar{F}(x)]^m - \frac{\gamma_r g_m(F(x))}{\bar{F}(x)}\right\}.$$ 

Thus,

$$\frac{C_{r-1}}{(r-1)!} \int_0^{\infty} x^{j+1}[\bar{F}(x)]^{\gamma_r} f(x) g_m^{r-1}(F(x))dx$$

$$= \frac{\beta^\alpha(j + 1)C_{r-1}}{\alpha \gamma_r(r - 1)!} \int_0^{\infty} x^{j+1}h'(x)dx. \tag{4.9}$$

Now integrating RHS in (4.9) by parts and using the value of $h(x)$ from (4.8), we get

$$\frac{C_{r-1}}{(r-1)!} \int_0^{\infty} x^{j+1}[\bar{F}(x)]^{\gamma_r} f(x) g_m^{r-1}(F(x))dx$$

$$= \frac{\beta^\alpha C_{r-1}}{\alpha (r - 1)!} \int_0^{\infty} x^{j-\alpha+1}[\bar{F}(x)]^{\gamma_r} g_m^{r-1}(F(x))dx.$$
which reduces to
\[
\frac{C_{r-1}}{(r-1)!} \int_0^\infty x^j [\bar{F}(x)]^{\gamma_r-1} g_m^{-1}(F(x)) \left\{ \bar{F}(x) - \frac{\beta^\alpha}{\alpha} x^{-\alpha+1} f(x) \right\} dx = 0. \quad (4.10)
\]

Now applying a generalization of the Müntz-Szász Theorem (Hwang and Lin, [7]) to equation (4.10), we get
\[
\frac{f(x)}{\bar{F}(x)} = \frac{\alpha}{\beta} \left( \frac{x}{\beta} \right)^{\alpha-1}
\]
which proves that
\[
\bar{F}(x) = e^{-(x/\beta)^\alpha}, \quad x > 0, \quad \alpha, \beta > 0.
\]

Acknowledgements
The authors is grateful to acknowledge with thanks to referees and Editor-in-Chief of the South Pacific Journal of Pure and Applied Mathematics for carefully reading the paper and for helpful suggestions which greatly improved the paper.

References


A Bootstrap Approach to Error-Reduction of Nonlinear Regression Parameters Estimation

H. O. Obiora-Ilouno
Department of statistics, Nnamdi Azikiwe University, Awka,
Anambra State, NIGERIA

J. I. Mbegbu
Department of Mathematics, University of Benin, Benin City,
Edo State, NIGERIA.
E-mail obiorailounoho@yahoo.com

Abstract

Bootstrap resampling, as computer intensive method has emerged as a powerful statistical tool for constructing inferential procedure in modern data analysis. In this paper, we proposed bootstrap algorithm for the estimation of parameters of the non-linear regression analysis. In estimating these parameters we adopted the Gauss-Newton method based on Taylor’s series to approximate the nonlinear regression model with the linear term, and subsequently employed least squares, bootstrap techniques to estimate the parameters iteratively. We used bootstrapping to provide estimates of exponential regression parameters. The computational difficulties that would have encountered in using the proposed method have been resolved by developing a computer program in R for the implementation of the algorithm.

Keywords: bootstrap, algorithm, exponential, regression, parameters, nonlinear, Gauss-Newton.

Introduction

Bootstrap resampling as computer intensive method has emerged as a powerful statistical tool for constructing inferential procedure in modern data analysis. The bootstrap approach introduced by Efron [3] was principally for the initial estimation of the distribution function of certain statistics. Since then, bootstrap has made tremendous impact in statistical theory. Bootstrapping, being a data-based simulation method, for statistical inference is used to study
the variability of estimated characteristics of the probability distribution of a set of observations and also, to provide confidence intervals for the population parameters (Efron [3], Efron [4]; Efron and Tibshirani [5]). It is a resampling technique in which sample of size n, are obtained with replacement from the original sample. The basic idea behind bootstrapping is to analyze the population by replacing the unknown distribution function \( F \) by the empirical distribution function \( \hat{F} \), obtained from the sample. (Efron, [4]; Mohammed [7]; Casella [2]). Exponential regression is a non-linear regression model, often used to measure the growth of a variable, such as population, GDP etc. An exponential relationship between \( X \) and \( Y \) exists whenever the dependent variable \( Y \) changes by a constant percentage, as the variable \( X \) also changes.

1 Material and Method

Given a model of the form

\[
Y = f(X_1, X_2, \ldots, X_k, \theta_1, \theta_2, \ldots, \theta_j) + \epsilon
\]

where the \( \theta \)'s are the parameters, \( X \)'s are the predictor variables and the error term \( \epsilon \sim N(0, \sigma^2) \) independently identically distributed and are uncorrelated. Equation (1) is assumed to be intrinsically nonlinear. Suppose we have a sample of \( n \) observations on the \( Y \) and \( X \)'s, then we can write

\[
Y_i = f(X_{i1}, X_{i2}, \ldots, X_{ik}, \theta_1, \theta_2, \ldots, \theta_j) + \epsilon_i; \quad i = 1, 2, \ldots, n
\]

The \( n \)-equation can be written compactly in a matrix notation as

\[
Y = f(X, \theta) + \epsilon
\]

where

\[
Y = \begin{bmatrix}
y_1 \\
y_2 \\
\vdots \\
y_n
\end{bmatrix},
X = \begin{bmatrix}
X_{11} & X_{12} & \cdots & X_{1k} \\
X_{12} & X_{22} & \cdots & X_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
X_{11} & X_{12} & \cdots & X_{kn}
\end{bmatrix},
\theta = \begin{bmatrix}
\theta_1 \\
\theta_2 \\
\vdots \\
\theta_j
\end{bmatrix},
\epsilon = \begin{bmatrix}
\epsilon_1 \\
\epsilon_2 \\
\vdots \\
\epsilon_n
\end{bmatrix}
\]

and \( E(\epsilon) = 0 \) The error sum of squares for the nonlinear model is defined as

\[
Q = S(\epsilon) = \sum_{i=1}^{n} [Y_i - f(X_i, \theta)]^2
\]

Denoting the least square estimates of \( \theta \) by \( \hat{\theta} \) these estimates minimize the \( S(\epsilon) \). The least square estimates of \( \theta \) are obtained by differentiating (4) with respect
to \( \theta \), equate to zero and solve for \( \hat{\theta} \), this results in \( j \) normal equations:

\[
\sum_{i=1}^{n} [Y_i - f(X_i, \hat{\theta})] \left[ \frac{\partial}{\partial \theta_p} f(X_i, \theta) \right]_{\theta=\hat{\theta}} ; \quad i = 1, 2, \ldots, n, p = 1, 2, \ldots, j
\]

In estimating the parameters of nonlinear regression model, we use the Gauss-Newton method based on Taylor’s series to approximate equation (3). Now, considering the function \( f(X, \theta) \) which is the deterministic component of

\[
Y_i = f(X_i, \theta) + \epsilon_i ; \quad i = 1, 2, \ldots, n
\]

Let \( \theta^0 \) be the initial approximate value of \( \theta \). Adopting Taylor’s series expansion of \( f(X_i, \theta) \) about \( \theta^0 \), we have the linear approximation

\[
f(X_i, \theta) = f(X_i, \theta^0) + (\theta - \theta^0) \left. \frac{\partial}{\partial \theta_p} f(X_i, \theta) \right|_{\theta=\theta^0}
\]

Substituting expressions (7) in (6) we obtain

\[
Y_i = f(X_i, \theta^0) + \sum_{p=1}^{j} \left. \frac{\partial}{\partial \theta_p} f(X_i, \theta) \right|_{\theta=\theta^0} (\theta_p - \theta^0_p) + \epsilon_i ; \quad i = 1, 2, \ldots, n, p = 1, 2, \ldots, j
\]

Equation (9) may be viewed as a linear approximation in a neighborhood of the starting value \( \theta^0 \)

Let \( f_i^0 = f(X_i, \theta^0) \), \( \beta_p^0 = \theta_p - \theta^0_p \)

\[
Z_{pi}^0 = \left. \frac{\partial}{\partial \theta_p} f(X_i, \theta) \right|_{\theta=\theta^0} , \quad i = 1, 2, \ldots, n, p = 1, 2, \ldots, j
\]

Hence, equation (9) becomes

\[
Y_i = f_i^0 + \sum_{p=1}^{j} Z_{pi}^0 \beta_p^0 + \epsilon_i , \quad i = 1, 2, \ldots, n
\]

\[
Y_i - f_i^0 = \sum_{p=1}^{j} Z_{pi}^0 \beta_p^0 + \epsilon_i , \quad i = 1, 2, \ldots, n
\]

In a matrix form, we have

\[
\begin{bmatrix}
Y_1 - f_1^0 \\
Y_2 - f_2^0 \\
\vdots \\
Y_n - f_n^0
\end{bmatrix} =
\begin{bmatrix}
Z_{11}^0 & Z_{12}^0 & \cdots & Z_{1j}^0 \\
Z_{21}^0 & Z_{22}^0 & \cdots & Z_{2j}^0 \\
\vdots & \vdots & \ddots & \vdots \\
Z_{nj}^0 & \cdots & \cdots & Z_{jn}^0
\end{bmatrix}
\begin{bmatrix}
\beta_1^0 \\
\beta_2^0 \\
\vdots \\
\beta_j^0
\end{bmatrix} +
\begin{bmatrix}
\epsilon_1 \\
\epsilon_2 \\
\vdots \\
\epsilon_n
\end{bmatrix}
\]
Compactly, equation (12) becomes

(13) \[ Y - f^0 = Z^0 \beta^0 + \epsilon \]

Where

\[ Y - f^0 = [Y_1 - f^0_1, Y_2 - f^0_2, \ldots, Y_n - f^0_n]' \]
\[ Z^0 = \begin{bmatrix} Z^0_{11} & \cdots & Z^0_{1j} \\ Z^0_{12} & \cdots & Z^0_{1j} \\ \vdots & \vdots & \vdots \\ Z^0_{nj} & \cdots & Z^0_{nj} \end{bmatrix}, \]
\[ \beta^0 = (\beta^0_1, \ldots, \beta^0_j), \epsilon = (\epsilon_1, \ldots, \epsilon_n) \]

We obtain the Sum of squares error \((SS\epsilon)\)

\[ SS\epsilon = (\epsilon' \epsilon) = ((Y - f^0) - Z^0 \beta^0)'((Y - f^0) - Z^0 \beta^0) \]
\[ = (Y - f^0)'(Y - f^0) - 2(Y - f^0)'(Z^0 \beta^0) + (Z^0 \beta^0)'(Z^0 \beta^0) \]
\[ \frac{\partial SS\epsilon}{\partial \beta^0} = 2(Y - f^0)Z^0 + 2(Z^0)'Z^0 = 0 \]

(14) \[ (Y - f^0)'Z^0 = Z^0(Z^0 \beta^0) \]

Hence

\[ \hat{\beta}^0 = (Y - f^0)'Z^0(Z^0'Z^0)^{-1} \]

Therefore, the least square estimates of \(\beta^0\) is

(15) \[ \hat{\beta}^0 = (Z^0'Z^0)^{-1}Z^0'(Y - f^0) \]

Thus, \(\hat{\beta}^0 = (\hat{\beta}^0_1, \hat{\beta}^0_2, \ldots, \hat{\beta}^0_j)'\) minimizes the error sum of squares,

(16) \[ S^*(\epsilon) = \sum_{i=1}^{n} \left( Y_i - f^0_i - \sum_{p=1}^{j} Z^0_{ip}\hat{\beta}^0_p \right)^2 \]

Now, the estimates of parameters \(\theta_p\) of non linear regression (1) are

(17) \[ \theta^1_p = \hat{\beta}^0_p + \theta^0_p; \quad p = 1, 2, \ldots, j \]

Iteratively, equation (17) reduces to

\[ \theta^1 = \hat{\beta}^0 + \theta^0 \]
\[ \theta^2 = \hat{\beta}^1 + \theta^1 \]
\[ \vdots \]
\[ \vdots \]
\[ \theta^r = \hat{\beta}^{r-1} + \theta^{r-1} \]
\[ \theta^{r+1} = \hat{\beta}^r + \theta^r \]

Thus

\[ \theta^{r+1} = \theta^r + (Z' Z)^{-1} Z' (Y - f^r) \]

where \( \beta^{r+1} = \theta^r + (Z' Z)^{-1} Z' (Y - f^r) \) are the least squares estimates of \( \beta \) obtained at the \((r + 1)\)th iterations. The iterative process continues until

\[ \left| \frac{\theta^{r+1} - \theta^r}{\theta^r} \right| < \delta \]

where \( \delta = 10^{-5} \) is the error tolerance ([8]). After each iteration, \( S^*(\epsilon) \) is evaluated to see if a reduction in its value has actually been achieved. At the end of the \((r + 1)\)th iteration, we have

\[ S^*(\epsilon)^r = \sum_{i=1}^{n} \left( Y_i - f_i^r - \sum_{p=1}^{j} Z_{pi} \hat{\beta}_p^r \right)^2 \]

and iteration is stopped if convergence is achieved. The final estimates of the parameters at the end of the \((r + 1)\)th iteration are:

\[ \theta^{r+1} = (\theta_1^{r+1}, \theta_2^{r+1}, \ldots, \theta_j^{r+1}) \]

2 The Bootstrap Algorithm based on the Resampling Observations for the Estimation of Non-linear Regression Parameters

Let \( W_i = (Y_i, Z_{ji})' \) be the original sample of size \( n \) for the resampling, and assume that are drawn independently and identically from a distribution of \( F \). Let \( Y_i = (y_1, y_2, \ldots, y_n)' \) be the column vector of the response variables, \( Z_{ji} = (z_{j1}, z_{j2}, \ldots, z_{jn})' \) is the matrix of dimension \( n \times k \) for the predictor variables, where \( j = 1, 2, \ldots, k \) and \( i = 1, 2, \ldots, n \). Let the vector \( (k \times 1) \times 1 \) vector \( W_i = (Y_i, Z_{ji})' \) denote the values associated with \( i^{th} (w_1, w_2, \ldots, w_n) \) observation sets.

Step 1:

Draw bootstrap sample \( (w_1^{(b)}, w_2^{(b)}, \ldots, w_n^{(b)}) \) of size \( n \) with replacement from the observation giving \( n^{-1} \) probability of each \( W_i \) value being sampled from the population and label the elements of each vector \( W_i^{(b)} = (Y_i^{(b)}, X_{ji}^{(b)})' \) where \( j = 1, 2, \ldots, k \) and \( i = 1, 2, \ldots, n \). From this form the vector for the response variable \( Y_i^{(b)} = (y_1^{(b)}, y_2^{(b)}, \ldots, y_n^{(b)}) \) and the matrix of the predictor variables \( Z_{ji}^{(b)} = (z_{j1}^{(b)}, z_{j2}^{(b)}, \ldots, z_{jn}^{(b)}) \)

Step 2:

Calculate the least square estimates for nonlinear regression coefficient from the bootstrap sample; \( \hat{\beta}^0 = (Z' Z)^{-1} Z (Y - f)' \)

Step 3:
Compute $\hat{\theta} = \hat{\theta}^0 + \hat{\beta}^0$ using the Gauss-Newton method, the $\hat{\theta}^1$ value is treated as the initial value in the first approximated linear model.

Step 4:
We return to the second step and again compute $\hat{\beta}'$s. At each iteration, new $\hat{\beta}'$s represent increments that are added to the estimates from the previous iteration according to step 3 and eventually find $\hat{\theta}^2$, which is $\hat{\theta} = \hat{\theta}^1 + \hat{\beta}'$. Step 5

Stopping Rule; the iteration process continues until $|\hat{\theta}^r + 1 - \hat{\theta}^r| < \delta$ where $\delta = 10^{-5}$, for the values of $\theta_0, \theta_1, \cdots, \theta_{p-1}$ from the first bootstrap sample $\hat{\theta}^{(b_1)}$.

Step 6:
Repeat steps 1 to 5 for $r = 1, 2, \cdots, B$, where $B$ is the number of repetition.

Step 7:
Obtain the probability distributions $F(\hat{\theta}^{(b)})$ of bootstrap estimates $\hat{\theta}^{(b_1)}, \hat{\theta}^{(b_2)}, \cdots, \hat{\theta}^{(b_n)}$ and use $(F(\hat{\theta}^{(b)})$ to estimate regression coefficients, variances. The bootstrap estimates of regression coefficient is the mean of the distribution $F(\hat{\theta}^{(b)}) ([10])$.

$$\hat{\theta}^{(b)} = \frac{1}{B} \sum_{i=1}^{n} \hat{\theta}^{(br)} = \tilde{\theta}^{(br)}$$

(21)

The bootstrap standard deviation from $F(\hat{\theta}^{(b)})$ distribution is

$$S(\hat{\theta}^{(b)}) = \sqrt{\frac{1}{B} \sum_{b=1}^{n} [(\hat{\theta}^{(br)} - \tilde{\theta}^{(br)})(\hat{\theta}^{(br)} - \tilde{\theta}^{(br)})']}^{\frac{1}{2}} ; r = 1, 2, \cdots, B$$

(22)

3 The Computer Program in R for Bootstrapping Non-Linear Regression

#x is the vector of independent variable
#theta is the vector of parameters of the model
#This function calculates the matrix of partial derivatives
F = function(x, theta)
{
output = matrix(0, ncol = 2, nrow = length(x))
for(i in 1 : length(x)) output[i,] = c(exp(theta[2]*x[i]), theta[1]*x[i]*exp(theta[2]*x[i]))
output
}

#This function calculates the regression coefficients using the Gauss –
Newton Method

```r
gaussnewton = function(y, x, initial, tol)
{
  theta = initial
  count = 0
  eps = y - (theta[1]*exp(theta[2]*x))
  SS = sum(eps**2)
  diff = 1
  while(tol < diff)
  {
    S = SS
    ff = F(x, theta)
    theta = c(theta + solve(t(ff)eps = y - (theta[1]*exp(theta[2]*x)))
    SS = sum(eps**2)
    diff = abs(SS - S)
    count = count + 1
    if(count == 100) break
    pp = c(theta, SS)
    # at each iteration
  }
  pp
}
```

This part of the code does the bootstrap

```r
boot = function(data, p, b, initial)
{
  n = length(data[, 1])
  z = matrix(0, ncol = p, nrow = n)
  output = matrix(0, ncol = p + 1, nrow = b)
  for(i in 1 : b)
  {
    u = sample(n, n, replace = T)
    for(j in 1 : n) z[j, ] = data[u[j], ]
    y = z[, 1]
    x = z[, 2 : p]
    logreg = gaussnewton(y, x, initial, 0.00001)
    coef = logreg
    output[i, ] = c(coef)
  }
  output
}
```

Then to run the code we use the following

```r
y < -c(data)
x < -c(data)
```
\[ \text{data} = \text{cbind}(y, x) \]
\[ \text{initial} = c(\text{initial}) \]
\[ \text{expo} = \text{boot}(\text{data}, p, B, \text{initial}) \]

# Run the following to view the bootstrap results
\[ \text{theta}_0 = \text{mean(expo[, 1])} \]
\[ \text{theta}_0 \]
\[ \text{theta}_1 = \text{mean(expo[, 2])} \]
\[ \text{theta}_1 \]
\[ \text{SSE} = \text{mean(expo[, 3])} \]
\[ \text{SSE} \]

3.1 Problem [Gujarati and Porter [6]]

The data below relates to the management fees that a leading mutual fund pays to its investment advisors, to manage its assets. The fees paid depends on the net asset value of the fund.

<table>
<thead>
<tr>
<th>Fee %</th>
<th>0.520</th>
<th>0.508</th>
<th>0.484</th>
<th>0.46</th>
<th>0.4398</th>
<th>0.4238</th>
<th>0.4115</th>
<th>0.402</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asset</td>
<td>0.5</td>
<td>5.0</td>
<td>10.0</td>
<td>15</td>
<td>20</td>
<td>25</td>
<td>30</td>
<td>35</td>
</tr>
<tr>
<td>Fee %</td>
<td>0.3944</td>
<td>0.388</td>
<td>0.3825</td>
<td>0.3738</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Asset</td>
<td>40</td>
<td>45</td>
<td>55</td>
<td>60</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 1: A developed a regression model for the management fees to the advisors.

4. Results and Discussion

From the data, the higher the net asset values of the fund, the lower are the advisory fees. Gujarati and Porter [6]

The Analytical Result of R Program (Without Bootstrapping)
\[ y < -c(0.520, 0.508, 0.484, 0.46, 0.4398, 0.4238, 0.4115, 0.402, 0.3944, 0.388, 0.3825, 0.3738) \]
\[ x < -c(0.5, 5, 10, 15, 20, 25, 30, 35, 40, 45, 55, 60) \]
\[ \text{run} = \text{gaussnewton}(y, x, c(0.5028, -0.002), 0.00001) \]
\[ \text{[1]}0.505805756 - 0.0054874790.001934099 \]
\[ \text{[1]}0.508724700 - 0.0059495980.001699222 \]
\[ \text{[1]}0.508897316 - 0.0059648110.001699048 \]

The estimated model from the results is:
\[ \hat{Y}_{ij} = 0.50890X^{-0.00596} \]

where Y and X are the fee and assets respectively
Relationship of advisory fees to fund assets
The Result of R Program Using Bootstrapping Algorithm

```r
> #run the code use the following for the bootstrap algorithm
> y <- c(0.520, 0.508, 0.484, 0.46, 0.4398, 0.4238, 0.4115, 0.402, 0.3944, 0.388, 0.3825, 0.3738)
> x <- c(0.5, 5, 10, 15, 20, 25, 30, 35, 40, 45, 55, 60)
> data = cbind(y, x)
> initial = c(0.5028, -0.002)
> expo = boot(data, 2, 1000, initial)
> #
> Run the following to view the bootstrap results
> theta_0 = mean(expo[, 1])
> theta_0
> [1] 0.5075696
> theta_1
> [1] -0.005964939
> SSE = mean(expo[, 3])
> SSE
> [1] 0.001328289
```

### 4.1 Discussion

\(\theta_0 = 0.50889\) and \(\theta_1 = 0.00596\) are the revised parameter estimates at the end of the last iteration. The least squares criterion measure \(SS\epsilon\) for the starting values has been reduced in the first iteration and also further reduced in second, third iteration respectively. The third iteration led to no change in either the estimates of the coefficient or the least squares \(SS\epsilon\) criterion measure. Hence, convergence is achieved, and the iteration end.

The fitted regression function is, \(\hat{Y} = 0.50889 \exp(-0.00596X)\) The results of the analytical and the bootstrap computations are shown in Table 2

<table>
<thead>
<tr>
<th></th>
<th>(\theta_0)</th>
<th>(\theta_1)</th>
<th>(SS\epsilon)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analytical</td>
<td>0.50889</td>
<td>-0.00596</td>
<td>0.001699</td>
</tr>
<tr>
<td>Bootstrap</td>
<td>0.50757</td>
<td>-0.00596</td>
<td>0.001328</td>
</tr>
</tbody>
</table>

Table 2: Analytical and Bootstrap Results.
The fitted regression function for both the analytical bootstrapping computation are
\( \hat{Y} = 0.50889 \exp(-0.00596X) \) and \( \hat{Y} = 0.50757 \exp(-0.00596X) \) respectively

4.2 Conclusion

The bootstrap approach yields approximately the same inference as the analytical method. The bootstrap algorithm yields a better reduced error sum of squares \( SS_\varepsilon \) than the analytical method. With these results, we have a greater confidence in the result obtained by bootstrap then the analytical result.

References


New Formula for Fractional Differentiation

Kuldeep Singh Gehlot
Department of Mathematics, Government Bangur College, Pali-306401, Rajasthan, India.
E-mail: drksgehlot@rediffmail.com

Jyotindra C. Prajapati
Charotar University of Science and Technology, Changa, Anand-388421, Gujarat, India.
E-mail: jyotindra18@rediffmail.com

Abstract

In this paper, authors introduced a new formula for Fractional Derivative in terms of Forward and Backward Differences. Authors also calculated Fractional Derivative of $x^n$, $\cos x$ and General Class of polynomial $S_m(x)$ with the help of newly defined formula.

Key Words: Forward Difference Operator, Backward Difference Operator, Fractional Derivative, Hypergeometric Function.


1 Introduction

1.1 Notations

Following notations used for deriving several results.
$\Delta_h = \text{Forward Difference Operator}$, $\nabla_h = \text{Backward Difference Operator}$, $D = \text{Differential Operator}$, $E = \text{Shift Operator}$, $I = \text{Identity Operator}$, $h = \text{Interval of Differences}$, $\mathbb{R} = \text{Set of Real Numbers}$ and $\mathbb{N} = \text{Set of Natural Numbers}$

1.2 Definitions
Let $t \in \mathbb{R}$ and $f(t)$ is a function of $t$ then for $n \in \mathbb{R}$, following Operators defined as:

**Shift Operator**

$$E^{nh}f(t) = f(t + nh), \quad E^{-jh}f(t) = f(t - jh)$$

**Forward Difference Operator**

$$\triangle_h f(t) = f(t + h) - f(t)$$

**Backward Difference Operator**

$$\nabla_h f(t) = f(t) - f(t - h)$$

**Differential Coefficient**

$$Df(t) = \lim_{h \to 0} \frac{f(t + h) - f(t)}{h}$$

### 1.3 Formulas

Well-known relationships between Shift Operator, Finite Differences and Differential Coefficient are given by

1. $$E^h \equiv e^{hD} \equiv I + \triangle_h$$

and

2. $$E^{-h} \equiv e^{-hD} \equiv I - \nabla_h$$

where $$D \equiv \frac{1}{h} \left[ \nabla_h + \frac{\nabla_h^2}{2} + \frac{\nabla_h^3}{3} - \ldots \right]$$

3. $$Df(t) = f^{(1)}(t) = \lim_{h \to 0} \frac{f(t + h) - f(t)}{h} = \lim_{h \to 0} \frac{\triangle_h f(t)}{h} = \lim_{h \to 0} \frac{\nabla_h f(t + h)}{h}$$

for higher order

4. $$D^{(n)}f(t) = f^{(n)}(t) = \lim_{h \to 0} \frac{\triangle^n_h f(t)}{h^n} = \lim_{h \to 0} \frac{\nabla^n_h f(t + nh)}{h^n}$$

5. $$\nabla^n_h f(t) = (I - E^{-h})^n f(t) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} C_j E^{-jh} f(t)$$

6. $$\nabla^n_h f(t) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} C_j e^{-jhD} f(t)$$
(7) $\nabla^n_h f(t) = \sum_{j=0}^{n} (-1)^j n C_j \sum_{i=0}^{\infty} \frac{(-hjD)^i}{(i)!} f(t)$

Formula for fractional order differences (CISM Lecture Notes [3]) defined as

(8) $\nabla^\alpha_h f(t) = \sum_{j=0}^{\infty} (-1)^j \alpha C_j E^{-jh} f(t)$

(9) $\nabla^\alpha_h f(t) = \sum_{j=0}^{\infty} (-1)^j \alpha C_j e^{-jhD} f(t)$

(10) $\nabla^\alpha_h f(t) = \sum_{j=0}^{\infty} (-1)^j \alpha C_j \sum_{i=0}^{\infty} \frac{(-hjD)^i}{(i)!} f(t)$

2 Main result

Result 1. The fractional forward and backward differences formula in terms of Derivatives for $\alpha \in \mathbb{R}^+$

(11) $\Delta^\alpha_h f(t) = (hD)^\alpha \sum_{j=0}^{\infty} \alpha C_j \left( \frac{hD}{2!} + \frac{h^2D^2}{3!} + \ldots \right)^j f(t)$

and

(12) $\nabla^\alpha_h f(t) = (hD)^\alpha \sum_{j=0}^{\infty} \alpha C_j \left( \frac{hD}{2!} - \frac{h^2D^2}{3!} + \ldots \right)^j f(t)$

Proof. For forward difference, from equation (1), we have

$$\Delta_h f(t) = (e^{hD} - I) f(t) = hD \left[ 1 + \frac{hD^2}{2} + \frac{h^2D^2}{3} + \ldots \right] f(t)$$

for $n^{th}$ difference, we get

$$\Delta^n_h f(t) = h^n D^n \left[ 1 + \left( \frac{hD^2}{2!} + \frac{h^2D^2}{3!} + \ldots \right) \right]^n f(t)$$

$$= h^n D^n \sum_{j=0}^{n} \frac{n C_j}{n!} \left( \frac{hD}{2} + \frac{h^2D^2}{3} + \ldots \right)^j f(t)$$

this formula can be generalized for fractional order differences (CISM Lecture Notes [3]) as

$$\Delta^\alpha_h f(t) = h^\alpha D^\alpha \sum_{j=0}^{\infty} \alpha C_j \left( \frac{hD}{2!} + \frac{h^2D^2}{3!} + \ldots \right)^j f(t).$$
Similarly, for backward difference, from equation (2), we have
\[ \nabla_h f(t) = (I - e^{-hD})f(t) \]
\[ = hD \left[ 1 - \frac{hD}{2!} + \frac{h^2D^2}{3!} - \ldots \right] f(t) \]

The \(n\)th difference gives
\[ \nabla^n_h f(t) = h^nD^n \left[ 1 - \left( \frac{hD}{2!} - \frac{h^2D^2}{3!} + \ldots \right)^n \right] f(t) \]
\[ = h^nD^n \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} \left( \frac{hD}{2!} - \frac{h^2D^2}{3!} + \ldots \right)^j f(t) \]

This formula can be generalized for fractional order differences (CISM Lecture Notes [3]) as
\[ \nabla^\alpha_h f(t) = h^\alpha D^\alpha \sum_{j=0}^{\infty} \frac{\alpha!}{j!(\alpha-j)!} \left( \frac{hD}{2!} - \frac{h^2D^2}{3!} + \ldots \right)^j f(t). \]

**Result 2.** Fractional Derivative formula in terms of Forward and Backward Differences are

(13) \[ D^\alpha f(t) = \lim_{h \to 0} \frac{1}{h^\alpha} \left( \sum_{j=0}^{\infty} \alpha C_j (-1)^j \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \nabla^i_h f(t + \alpha h) \right) \]

(14) \[ D^\alpha f(t) = \lim_{h \to 0} \frac{1}{h^\alpha} \left( \sum_{j=0}^{\infty} \alpha C_j (-1)^j \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \nabla^i_h f(t + \alpha h) \right) \]

Another forms

(15) \[ D^\alpha f(t) = \frac{\nabla^\alpha_h (-1)^{2\alpha} \sum_{j=0}^{\infty} \alpha C_j \left( \frac{\nabla^2_h}{2!} + \frac{\nabla^3_h}{3!} + \ldots \right)^j f(t) \]

(16) \[ D^\alpha f(t) = \frac{\nabla^\alpha_h (-1)^{2\alpha} \sum_{j=0}^{\infty} \alpha C_j \left( \frac{\nabla^2_h}{2!} + \frac{\nabla^3_h}{3!} + \ldots \right)^j f(t) \]

**Proof** From equation (4) and (5), we have
\[ D^\alpha f(t) = \lim_{h \to 0} \frac{1}{h^\alpha} \left( \sum_{j=0}^{n} \alpha C_j (-1)^j E^{-jh} f(t + nh) \right) \]

This formula can be generalized for fractional order derivatives (CISM Lecture Notes [3]) as
\[ (17) \quad D^\alpha f(t) = \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{j=0}^{\infty} \alpha C_j(-1)^j E^{-jh} f(t + \alpha h) \]

from (1), we have

\[ (18) \quad E^{-jh} \equiv (I + \triangle_h)^{-j} \equiv \sum_{i=0}^{\infty} (-1)^i \alpha C_i \triangle_h^i \]

and

\[ (19) \quad E^{-jh} \equiv (I - \nabla_h)^j \equiv \sum_{i=0}^{\infty} j C_i (-1)^i \nabla_h^i \]

from (17) and (18), we get

\[ D^\alpha f(t) = \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{j=0}^{\infty} \alpha C_j(-1)^j \sum_{i=0}^{\infty} (-1)^i \alpha C_i \triangle_h^i f(t + \alpha h) \]

This completes the proof of (13).

Equations (17) and (19) leads to

\[ D^\alpha f(t) = \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{j=0}^{\infty} \alpha C_j(-1)^j \sum_{i=0}^{\infty} j C_i (-1)^i \nabla_h^i f(t + \alpha h) \]

This completes the proof of (14).

Again from (1), we have

\[ D^n f(t) = \frac{1}{h^n} \left[ \triangle_h - \frac{\triangle_h^2}{2} + \frac{\triangle_h^3}{3} - \ldots \right]^n f(t) \]

using Binomial expansion, we obtain

\[ D^n f(t) = \frac{\triangle_h^n}{h^n} \sum_{j=0}^{n} n C_j (-1)^j \left( \frac{\triangle_h}{2} - \frac{\triangle_h^2}{3} + \ldots \right)^j f(t), \]

This formula can also be generalized to the case of fractional order derivatives (CISM Lecture Notes [3]) as

\[ D^\alpha f(t) = \frac{\triangle_h^\alpha}{h^\alpha} \sum_{j=0}^{\infty} \alpha C_j(-1)^j \left( \frac{\triangle_h}{2} - \frac{\triangle_h^2}{3} + \ldots \right)^j f(t). \]

This completes the proof of (15).

Again from (2), we have

\[ D^n f(t) = \frac{(-1)^n}{h^n} \left[ - \nabla_h - \frac{\nabla_h^2}{2} - \frac{\nabla_h^3}{3} - \ldots \right]^n f(t) \]
using Binomial expansion, we obtain

\[ D^n f(t) = \frac{(-1)^{2n} h^n}{h^n} \sum_{j=0}^{\infty} C_j \left( \frac{h}{2} + \frac{h^2}{3} + \ldots \right)^j f(t), \]

This formula can be generalized to the case of fractional order derivatives (CISM Lecture Notes [3]) as

\[ D^\alpha f(t) = \frac{(-1)^{2\alpha} h^n}{h^{\alpha}} \sum_{j=0}^{\infty} C_j \left( \frac{h}{2} + \frac{h^2}{3} + \ldots \right)^j f(t). \]

This completes the proof of (16).

**Result 3.** The Fractional derivative of \( x^n \) is given by

(20) \( D^\alpha (x^n) = \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{j=0}^{\infty} C_j (-1)^j E^{-jh} (x + \alpha h)^n \)

where \( \alpha \leq n \).

**Proof** From equation (17), we have

(21) \( D^\alpha (x^n) = \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{j=0}^{\infty} C_j (-1)^j e^{-jh} (x + \alpha h)^n \),

using (2), we obtain

\[ D^\alpha (x^n) = \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{j=0}^{\infty} C_j (-1)^j e^{-jhD} (x + \alpha h)^n \]

\[ = \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{j=0}^{\infty} C_j (-1)^j \sum_{i=0}^{\infty} \frac{(-jhD)^i}{(i)!} (x + \alpha h)^n \]

this equation reduces to,

(22) \( D^\alpha (x^n) = \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{j=0}^{\infty} C_j (-1)^j \sum_{i=0}^{\infty} \frac{(-1)^j (hj)^i (n)!}{(i)! (n-i)!} (x + \alpha h)^n \)

The following result (23) mentioned in (Erdelyi et al [1], page 85)
\[ \alpha C_j = \frac{(-1)^j \Gamma(j-\alpha)}{\Gamma(j+1)\Gamma(-\alpha)} \]

From (22) and (23), we obtain
\[
D^\alpha(x^n) = \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{j=0}^{\infty} \frac{(-\alpha)_j (x + \alpha h)^n}{(j)!} \sum_{i=0}^{\infty} \frac{(-n)_i}{(i)!} \left( \frac{jh}{x+\alpha h} \right)^i,
\]
\[= \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{j=0}^{\infty} \frac{(-\alpha)_j (x + \alpha h)^n}{(j)!} {}_1F_0 \left[ -n; -; \frac{jh}{x+\alpha h} \right] \]

**Result 4.** The Fractional derivative of \( \cos x \) is given by
\[
D^\alpha \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{j=0}^{\infty} \frac{(-\alpha)_j (x + \alpha h)^{2k}}{(j)!} {}_1F_0 \left[ -2k; -; \frac{jh}{x+\alpha h} \right]
\]

**Result 5.** The Fractional derivative of \( S^m_n(x) \) a general class of polynomial is given by
\[
D^\alpha S^m_n(x) = \sum_{k=0}^{\infty} \frac{(-n)_m k}{(k)!} \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{j=0}^{\infty} \frac{(-\alpha)_j (x + \alpha h)^k}{(j)!} {}_1F_0 \left[ -k; -; \frac{jh}{x+\alpha h} \right]
\]

**Proof.** The general class of polynomial given by (Srivastava [2]), is
\[
S^m_n(x) = \sum_{k=0}^{\infty} \frac{(-n)_m k}{(k)!} A_{m,k} x^k
\]
where \( m \) is the arbitrary positive integer, the coefficient \( A_{m,k}; n, k > 0 \) are arbitrary constant real or complex.
Using the result of theorem 2 and equation (26), we immediately get the desire result.

**Note:** Thus, we can easily obtain fractional derivatives of all functions and polynomials which are in power forms.

**References**


[3] CISM Lecturer Notes, Published in Fractals and Fractional Calculus in
Compromise Allocation in Multivariate Stratified Sampling: A Stochastic Programming Approach

Ummatul Fatima, Rahul Varshney and M.J.Ahsan
Department of Statistics & Operations Research
Aligarh Muslim University Aligarh (INDIA).

Abstract

In this paper the problem of optimum allocation in multivariate stratified surveys is studied as a Multiobjective Integer Nonlinear Stochastic Programming Problem. Assuming the probability distribution of the random coefficients, the problem is converted into its deterministic equivalent. A solution procedure is developed using Goal Programming Technique. A numerical example is presented to illustrate the application of the proposed solution procedure. The solution obtained is then compared with some well known compromise allocations to show that the proposed procedure gives more precise results. The numerical results are obtained by using the optimization software LINGO.

Keywords Multivariate Stratified Surveys, Compromise Allocation, Multiobjective Integer Nonlinear Stochastic Programming, Goal Programming Technique.

1 Introduction


Most of the authors who discussed this problem treated it as a deterministic problem. In actual practice, if the true strata variances are not known, then the problem turns out to be a Multiobjective Stochastic Integer Nonlinear Programming Problem (MSINLPP). This is because of the fact that the sample

\(^2\text{Corresponding author. Email: fatimau2011@yahoo.com}\)
estimates of the strata variances are random variables instead of deterministic constants as assumed in the classical stratified sampling.

This paper deals with the problem of obtaining a compromise allocation for multivariate stratified sampling as a MSINLPP. A method, using Goal Programming is proposed for obtaining the said compromise allocation. A numerical example is also presented to illustrate the computational details. In section 6 of the manuscript a comparative study has been made to compare the performance of the proposed compromise allocation with that of the proportional and Cochran’s average allocations.

2 The Multiobjective Stochastic Integer Nonlinear Programming Problem (MSINLPP)

Consider a population of size \( N \) divided into \( L \) non-overlapping and exhaustive strata, of sizes \( N_1, N_2, \ldots, N_h, \ldots, N_L \) with \( \sum_{h=1}^{L} N_h = N \). Independent simple random samples of sizes \( n_h; h = 1, 2, \ldots, L \), are drawn, without replacement, from each of the \( L \) strata. Let \( p \geq 2 \), characteristics be defined on each population unit and the estimation of the \( p \) - population means \( \bar{Y}_j; j = 1, 2, \ldots, p \), is of interest. Unless specified otherwise the notations of Cochran (1977) are used.

It is known that in stratified sample surveys the stratified sample mean \( \bar{y}_{st} = \sum_{h=1}^{L} W_h \bar{y}_h \) is an unbiased estimate of the population mean mean \( \bar{Y} \) with a sampling variance

\[
V(\bar{y}_{st}) = \sum_{h=1}^{L} \frac{W_h^2 S^2_h}{n_h} - \sum_{h=1}^{L} \frac{W_h^2 S^2_h}{N_h}
\]  

(2.1)

For a multivariate population with \( p \) - characteristics under study suffix “\( j \)” may be introduced to denote the \( j^{th} \) characteristic. Thus \( \bar{y}_{st} = \sum_{h=1}^{L} W_h \bar{y}_{hj} \) will be an unbiased estimate of the population mean \( \bar{Y}_j \) of the \( j^{th} \) characteristics with a sampling variance

\[
V(\bar{y}_{jst}) = \sum_{h=1}^{L} \frac{W_h^2 S^2_{hj}}{n_{hj}} - \sum_{h=1}^{L} \frac{W_h^2 S^2_{hj}}{N_h}; j = 1, 2, \ldots, p
\]  

(2.2)

where

\[
S^2_{hj} = \frac{1}{N_h - 1} \sum_{i=1}^{N_h} (y_{hij} - \bar{Y}_j)^2
\]  

(2.3)

is the stratum variance for the \( h^{th} \) stratum of the \( j^{th} \) characteristic; \( h = \)
1, 2, ..., L; \( j = 1, 2, ..., p \). In univariate stratified surveys the problem of optimum allocation involves the determination of the sample sizes \( n_h; h = 1, 2, ..., L \) that minimize the variance \( V(\bar{y}_{st}) \) within the available budget or alternatively minimize the total cost of the survey for a fixed precision of the estimator. Here-in-after we discussed the former case only because in most of the practical situations the budget is fixed in advance. The total cost \( C \) of a stratified sample survey is usually expressed as a linear function of the sample sizes \( n_h \) as \( C = c_0 + \sum_{h=1}^{L} c_h n_h \). If the cost of the travelling between the units selected in the sample from a given stratum is also significant then the linear cost function will not be an adequate approximation to the actual cost incurred. Beardwood et al. (1959) suggested that the cost of visiting the \( n_h \) randomly selected units in the \( h^{th} \) stratum may be taken as \( t_h \sqrt{n_h}; h = 1, 2, ..., L \) approximately, where \( t_h \) is the travel cost per unit in the stratum (Cochran (1977)). This conjecture is based on the fact that the distance between \( k \) randomly scattered points is proportional to \( \sqrt{k} \).

Under the above circumstance the cost function may be given as

\[
C = c_0 + \sum_{h=1}^{L} c_h n_h + \sum_{h=1}^{L} t_h \sqrt{n_h}
\]  

(2.4)

It can be seen that the RHS of (2.4) is quadratic in \( \sqrt{n_h} \).

Thus the problem of optimum allocation in a univariate stratified survey with a significant travel cost between selected units within strata may be expressed as the following Integer Nonlinear Programming Problem (INLPP)

Minimize

\[
\sum_{h=1}^{L} \frac{W_h^2 S_h^2}{n_h} - \sum_{h=1}^{L} \frac{W_h^2 S_h^2}{N_h}
\]

subject to

\[
c_0 + \sum_{h=1}^{L} c_h n_h + \sum_{h=1}^{L} t_h \sqrt{n_h} \leq A
\]

(2.5)

\[
2 \leq n_h \leq N_h
\]

and

\( n_h \) integers; \( h = 1, 2, ..., L \)

where ‘\( A \)’ is the available budget.

The restrictions \( 2 \leq n_h \leq N_h; h = 1, 2, ..., L \) are introduced to obtain the estimates of the stratum variances and to avoid the problem of oversampling. The integer restrictions are imposed because for practical implementation of the allocations, integer values of the sample sizes are required.
Usually the stratum variances $S^2_h$ are unknown and their sample estimates

$$s^2_h = \frac{1}{n_h - 1} \sum_{i=1}^{n_h} (y_{hi} - \bar{y}_h)^2$$  \hspace{1cm} (2.6)

are used. This gives the INLPP (2.5) as:

Minimize $\hat{V} = \sum_{h=1}^{L} W^2_h s^2_h n_h - \sum_{h=1}^{L} W^2_h s^2_h N_h$

subject to $c_0 + \sum_{h=1}^{L} c_h n_h + \sum_{h=1}^{L} t_h \sqrt{n_h} \leq C_0$

$$2 \leq n_h \leq N_h$$

and $n_h$ integers; $h = 1, 2, ..., L$

where $C_0 = A - c_0$ denotes the amount available for to meet the measurement and travel costs, and $\hat{V}$ denote the estimated value of the variance $V(\bar{y}_{st})$. The sample stratum variance $s^2_h$ for a fixed $h$ is a random variable (Díaz-García and Garay-Tápia (2007)) because it is a sample statistic. Furthermore, although the budget available for the survey ‘$A$’ is fixed in advance, the per unit cost of measurement $c_h$ and the per unit travelling cost $t_h$ may vary during the execution time of the survey due to random causes. Therefore, the costs $c_h$ and $t_h$ are also random variables. Thus in actual practice the INLPP (2.7) turns out to be a stochastic integer nonlinear programming problem (SINLPP) where $s^2_h, c_h$ and $t_h$ as random variables.

While dealing with a multivariate stratified population for obtaining the individual optimum allocation for the $j^{th}$ characteristic the cost $C_0$ available for measurement and travel must be allocated characterwise. (See Jahan et al. (1994)). Let $C_{0j}; j = 1, 2, ..., p$ with $\sum_{j=1}^{p} C_{0j} = C_0$, be the amount allocated for measurement of the $j^{th}$ characteristic and the associated travel cost. Thus the individual optimum allocations will be the solution to the following SINLPP. The suffix “$j$” has been introduced to denote the $j^{th}$ characteristic.

Minimize $\hat{V}_j = \sum_{h=1}^{L} W^2_h s^2_{hj} n_{hj} - \sum_{h=1}^{L} W^2_h s^2_{hj} N_h$

subject to $\sum_{h=1}^{L} c_{hj} n_{hj} + \sum_{h=1}^{L} t_h \sqrt{n_{hj}} \leq C_{0j}$

$$2 \leq n_{hj} \leq N_{hj}$$

and $n_{hj}$ integers; $h = 1, 2, ..., L$.
2 \leq n_{hj} \leq N_h
\text{and} \quad n_{hj} \text{ integers; } h = 1, 2, ..., L

with \( s_{hj}^2, c_{hj} \) and \( t_h \) as random variables, where \( c_{hj} \) is the per unit cost of measurement in the \( h^{th} \) stratum for the \( j^{th} \) characteristic. Thus for \( j = 1, 2, ..., p \) we have \( p \) separate SINLPP's, defined by (2.8). Unfortunately the individual optimum allocations are of no practical use because usually they vary widely from characteristic to characteristic.

In univariate case when only \( s_h^2 \) are random variables Díaz García and Garay-Tápia (2007) worked out the optimum allocation by converting the problem into its deterministic equivalent. But their approach was not free from problems. They presented three stochastic models, the E-model, the Modified E-model and V-model, but they did not show which one of these approaches is the best under any given circumstances. Kozak and Wang (2010) also tried to identify the best allocation among those presented by Díaz García and Garay-Tápia (2007). They compared these allocations with the classical allocation and carried out a simulation study. Unfortunately they too concluded with the following remark.

"From our results it follows that at the moment we cannot claim that stochastic programming offers allocation that would perform better than the classical allocation method".

Some others who discussed the multivariate allocation problem as a stochastic programming problem are: Khan et al. (2011), they used Chebyshev’s approximation to linearize the cost constraint with random measurement cost in a multivariate stratified survey. Ghufran et al. (2011) formulated the problem as a Multiobjective Nonlinear Programming Problem with random measurement and travel costs but known deterministic values of true stratum variances \( S_{hj}^2 \).

In the present paper the problem of obtaining an optimum compromise allocation in multivariate stratified surveys with \( s_{hj}^2, c_{hj} \) and \( t_h \) as random variables is treated as a MSINLPP where the true stratum variances \( S_{hj}^2 \) are unknown. The mathematical model of the formulated MSINLPP may be given as

\[
\begin{bmatrix}
\hat{V}_1 \\
\hat{V}_2 \\
\vdots \\
\hat{V}_j \\
\vdots \\
\hat{V}_p
\end{bmatrix}
\]

Minimize \( \sum_{h=1}^{L} c_{hj} \hat{n}_{h(c)} + \sum_{h=1}^{L} t_h \sqrt{\hat{n}_{h(c)}} \leq C_{0j}; j = 1, 2, ..., p \) (2.9)
\[ 2 \leq n_{h(c)} \leq N_h \]

and

\[ n_{h(c)} \text{ integers; } h = 1, 2, ..., L \]

where \( n_{h(c)}; h = 1, 2, ..., L \) denote the required compromise allocation.

In the next section MSINLPP (2.9) is converted into a deterministic problem.

### 3 Conversion of the MSINLPP in deterministic form

First the SINLPP (2.8) is converted into its equivalent deterministic form for a fixed \( j \). Consider the objective function of the SINLPP (2.8) for \( j^{th} \) characteristic, that is,

\[ \hat{V}_j = \sum_{h=1}^{L} \frac{W_h^2 s^2_{hj}}{n_{hj}} - \sum_{h=1}^{L} \frac{W_h^2 s^2_{hj}}{N_h} \] (3.1)

where the sample variances \( s^2_{hj} \) are random variables. Melaku (1986) showed that the quantity

\[ \gamma_{hj} = \frac{1}{n_{hj} - 1} \sum_{i=1}^{n_{hj}} (y_{hij} - \bar{Y}_{hj})^2 \] (3.2)

has an asymptotically normal distribution with mean

\[ E(\gamma_{hj}) = \frac{n_{hj}}{n_{hj} - 1} S^2_{hj} \] (3.3)

and variance

\[ V(\gamma_{hj}) = \frac{n_{hj}}{(n_{hj} - 1)^2} [C^4_{hj} - (S^2_{hj})^2] \] (3.4)

where

\[ C^4_{hj} = \frac{1}{N_h} \sum_{i=1}^{N_h} (y_{hij} - \bar{Y}_{hj})^4 \] (3.5)

is the fourth moment about the stratum mean \( \bar{Y}_{hj} \).

Now

\[ \gamma_{hj} - \frac{n_{hj}}{n_{hj} - 1} (\bar{y}_{hj} - \bar{Y}_{hj})^2 \]

\[ = \frac{1}{n_{hj} - 1} \sum_{i=1}^{n_{hj}} (y_{hij} - \bar{Y}_{hj})^2 - \frac{n_{hj}}{n_{hj} - 1} \bar{y}_{hj}^2 - \frac{n_{hj} - 1}{n_{hj} - 1} \bar{Y}_{hj}^2 + 2 \frac{n_{hj}}{n_{hj} - 1} \bar{y}_{hj} \bar{Y}_{hj} \]

\[ = \frac{1}{n_{hj} - 1} \sum_{i=1}^{n_{hj}} y_{hij}^2 - \frac{n_{hj}}{n_{hj} - 1} \bar{y}_{hj}^2 \text{ (on simplification)} \] (3.6)
Again \( s_{hj}^2 = \frac{1}{n_{hj}-1} \sum_{i=1}^{n_{hj}} (y_{hij} - \bar{y}_{hj})^2 \)

\[
= \frac{1}{n_{hj}-1} \sum_{i=1}^{n_{hj}} y_{hij}^2 - \frac{n_{hj}}{n_{hj}-1} \bar{y}_{hj}^2
\]

(3.7)

(3.6) and (3.7) \( \Rightarrow \)

\[
s_{hj}^2 = \gamma_{hj} - \frac{n_{hj}}{n_{hj}-1} (\bar{y}_{hj} - \bar{Y}_{hj})^2
\]

(3.8)

As \( \frac{n_{hj}}{n_{hj}-1} \to 1 \) and \( \bar{y}_{hj} - \bar{Y}_{hj} \to 0 \) in probability \( s_{hj}^2 \to \gamma_{hj} \) asymptotically.

\( \Rightarrow \) \( s_{hj}^2 \) have an asymptotical independent normal distribution with mean \( E(\gamma_{hj}) \) and variance \( V(\gamma_{hj}) = \frac{n_{hj}}{(n_{hj}-1)^2} [C_{hj}^4 - (S_{hj}^2)^2] \) for \( h = 1, 2, ..., L; j = 1, 2, ..., p. \)

Extending the result of Díaz-García and Garay-Tápia (2007) for multivariate case, the \( j^{th} \) estimated variance \( \hat{V}_j; j = 1, 2, ..., p. \) in the objective function of SINLPP (2.8), being a linear combination of independently and normally distributed random variables, \( s_{hj}^2 \) can also be treated as having independent normal distributions with means

\[
E(\hat{V}_j) = \sum_{h=1}^{L} \frac{W_h^2 s_{hj}^2}{n_{hj}-1} - \sum_{h=1}^{L} \frac{W_h^2}{N_h} \left( \frac{n_{hj}}{n_{hj}-1} \right) S_{hj}^2; j = 1, 2, ..., p
\]

(3.9)

and variances

\[
V(\hat{V}_j) = \sum_{h=1}^{L} \frac{W_h^4}{n_{hj}(n_{hj}-1)^2} \left( C_{hj}^4 - (S_{hj}^2)^2 \right) \\
- \sum_{h=1}^{L} \frac{W_h^2}{N_h^2} \left[ \frac{n_{hj}}{(n_{hj}-1)^2} \left( C_{hj}^4 - (S_{hj}^2)^2 \right) \right]; j = 1, 2, ..., p
\]

(3.10)

Thus the deterministic equivalent of the objective function of SINLPP (2.8) may be expressed as a function \( \hat{\phi}_j() \) of the sample sizes \( n_{hj} \), that is,

\[
\hat{\phi}_j(n_{1j}, n_{2j}, ..., n_{hj}, ..., n_{Lj}) = \alpha_1 E(\hat{V}_j) + \alpha_2 \sqrt{V(\hat{V}_j)}
\]

(3.11)

where \( E(\hat{V}_j) \) and \( V(\hat{V}_j) \) are given by (3.9) and (3.10) respectively and \( \alpha_1, \alpha_2 \geq 0 \) are known constants representing the relative importance of \( E(\hat{V}_j) \) and \( V(\hat{V}_j) \).

Without loss of generality we can assume that \( \alpha_1 + \alpha_2 = 1. \)

Now consider the \( j^{th} \) probabilistic constraint in SINLPP (2.8)

\[
\sum_{h=1}^{L} c_{hj} n_{hj} + \sum_{h=1}^{L} t_h \sqrt{n_{hj}} \leq C_{0j}
\]

(3.12)
Assume that the cost $c_{hj}$ and $t_h$ are independently and normally distributed as $N(\mu_{c_{hj}}, \sigma^2_{c_{hj}})$ and $N(\mu_{t_h}, \sigma^2_{t_h})$ respectively. Being a linear combination of independent normal distributions LHS in (3.12) will also be a normally distributed random variable. Usually the parameters $\mu_{c_{hj}}, \mu_{t_h}$ and $\sigma^2_{c_{hj}}, \sigma^2_{t_h}$ are unknown but they can be estimated from a pilot survey or their values at some previous occasion may be used. If $\hat{c}_{hj}$, $\hat{t}_h$ and $\hat{\sigma}^2_{c_{hj}}, \hat{\sigma}^2_{t_h}$ are the estimated values of $\mu_{c_{hj}}, \mu_{t_h}$ and $\sigma^2_{c_{hj}}, \sigma^2_{t_h}$ respectively, then the usual deterministic equivalent of the LHS of (3.12) may be given as

$$
\beta_1 \left( \sum_{h=1}^{L} \hat{c}_{hj} n_{hj} + \sum_{h=1}^{L} \hat{t}_h \sqrt{n_{hj}} \right) + \beta_2 \left( \sqrt{\sum_{h=1}^{L} n_{hj}^2 \hat{\sigma}^2_{c_{hj}} + \sum_{h=1}^{L} n_{hj} \hat{\sigma}^2_{t_h} } \right) = (3.13)
$$

In (3.13) the quantities inside the first and second () represent $E(\hat{C}_{oj})$ and $V(\hat{C}_{oj})$ respectively. Where $\hat{C}_{oj} = \sum_{h=1}^{L} \hat{c}_{hj} n_{hj} + \sum_{h=1}^{L} \hat{t}_h \sqrt{n_{hj}}; j = 1, 2, ..., p$.

$\beta_1, \beta_2 \geq 0$ are known constants representing the relative importance of the terms $E(\hat{C}_{oj})$ and $V(\hat{C}_{oj})$ respectively. Since sampling is independent in each stratum the cross product terms in $V(\hat{C}_{oj})$ vanish. Without loss of generality we can assume that $\beta_1 + \beta_2 = 1$. Thus the deterministic equivalent of the cost constraint (3.12) may be given as

$$
\beta_1 \left( \sum_{h=1}^{L} \hat{c}_{hj} n_{hj} + \sum_{h=1}^{L} \hat{t}_h \sqrt{n_{hj}} \right) + \beta_2 \left( \sqrt{\sum_{h=1}^{L} n_{hj}^2 \hat{\sigma}^2_{c_{hj}} + \sum_{h=1}^{L} n_{hj} \hat{\sigma}^2_{t_h} } \right) \leq c_{oj} \quad (3.14)
$$

The expressions (3.9), (3.10), (3.11) and (3.14) give the deterministic equivalent of the SINLPP (2.8) as the following All Integer Nonlinear Programming Problem (AINLPP):

Minimize $\hat{\phi}_j(n_1, n_2, ..., n_{hj}, ..., n_{Lj}) = \alpha_1 \left[ \sum_{h=1}^{L} \frac{W^2 s_{hj}^2}{n_{hj} - 1} - \sum_{h=1}^{L} \frac{W^2}{N_h} \left( \frac{n_{hj}}{n_{hj} - 1} \right) s_{hj}^2 \right]$

$$
+ \alpha_2 \left\{ \left( \sum_{h=1}^{L} \frac{W^4}{n_{hj} (n_{hj} - 1)^2 \left( C_{hj}^4 - (s_{hj}^2)^2 \right) } \right) - \left( \sum_{h=1}^{L} \frac{W^2}{N_h^2} \left( \frac{n_{hj}}{n_{hj} - 1} \right)^2 \left( C_{hj}^4 - (s_{hj}^2)^2 \right) \right) \right\}^{1/2}
$$

subject to $\beta_1 \left( \sum_{h=1}^{L} \hat{c}_{hj} n_{hj} + \sum_{h=1}^{L} \hat{t}_h \sqrt{n_{hj}} \right) + \beta_2 \left( \sqrt{\sum_{h=1}^{L} n_{hj}^2 \hat{\sigma}^2_{c_{hj}} + \sum_{h=1}^{L} n_{hj} \hat{\sigma}^2_{t_h} } \right) \leq c_{oj}$

$$
2 \leq n_{hj} \leq N_h \quad \text{for } n_{hj} \text{ integers; } h = 1, 2, ..., L \quad (3.15)
$$
Note that (3.15) defines $p$ separate problems for $j = 1, 2, ..., p$.

The deterministic equivalent of the MSINLPP (2.9) can now be given as the following MINLPP

\[
\begin{align*}
\text{Minimize} & \quad \begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \\ \vdots \\ \hat{\phi}_j \\ \vdots \\ \hat{\phi}_p \end{bmatrix} \\
\text{subject to} & \quad \beta_1 \left( \sum_{h=1}^{L} c_{h} n_{h(c)} + \sum_{h=1}^{L} \bar{t}_h \sqrt{n_{h(c)}} \right) + \beta_2 \left( \sqrt{\sum_{h=1}^{L} n_{h(c)}^2 \hat{\sigma}_{c_h}^2 + \sum_{h=1}^{L} n_{h(c)} \hat{\sigma}_{t_h}^2} \right) \leq C_0 \\
& \quad 2 \leq n_{h(c)} \leq N_h \\
& \quad n_{h(c)} \text{ integers}; \, h = 1, 2, ..., L 
\end{align*}
\]

where $\hat{\phi}_j = \hat{\phi}_j (n_{1j}, n_{2j}, ..., n_{hj}, ..., n_{Lj}); \, j = 1, 2, ..., p$ (3.17)

are as given by the objective function of (3.15) and $n_{h(c)}; \, h = 1, 2, ..., L$ denote the required compromise allocation. Also in the cost constraint now we have $c_h = \sum_{j=1}^{p} c_{hj}$, that is $c_h$ is the cost of measuring all the $p$ - characteristics on a selected unit from the $h^{th}$ stratum. Since $c_{hj}$ are independent ‘$c_h$’ being the sum of independent normal variates will also be a normal variate $N(\mu_{c_h}, \sigma_{c_h}^2)$. If $\mu_{c_h}$ and $\sigma_{c_h}^2$ are unknown their sample estimates $\bar{c}_h$ and $\hat{\sigma}_{c_h}^2$, respectively, for $h^{th}$ stratum may be used. After converting the formulated MSINLPP (2.9) into its deterministic equivalent in the second phase, the deterministic multiobjective programming problem is converted into a single objective Goal Programming Problem using a suitable compromise criterion. When numerical values of the coefficients are available the Goal Programming Problem may be solved, using the optimization software LINGO (2001).

4 The goal programming approach

Let $\phi_j^*$ be the optimum value of $\phi_j$ defined in (3.15) under the optimum allocation of $j^{th}$ characteristic obtained by solving the following AINLPP for each $j = 1, 2, ..., p$ separately.
Minimize $\hat{\phi}_j$

subject to $\beta_1 \left( \sum_{h=1}^{L} \bar{c}_{hj} n_{hj} + \sum_{h=1}^{L} \bar{t}_{hj} \sqrt{n_{hj}} \right) + \beta_2 \left( \sum_{h=1}^{L} n_{hj}^2 \hat{\sigma}_{chj}^2 + \sum_{h=1}^{L} n_{hj} \hat{\sigma}_{thj}^2 \right) \leq C_{0j}$

\[ \text{(4.1)} \]

where $\hat{\phi}_j; j = 1, 2, \ldots, p$ are as given in (3.17).

The solution to (4.1) will be the individual optimum allocation $n^*_j = (n^*_{1j}, n^*_{2j}, \ldots, n^*_{Lj}); j = 1, 2, \ldots, p$ under the stated conditions. To use the ‘Goal Programming Approach’ to solve the MINLPP (3.16), first we convert it into a single objective problem using a suitable compromise criterion.

Let $\tilde{\phi}_j$ denote the value of $\hat{\phi}_j$ for a compromise allocation $n_{hj}(c)$. As $\phi^*_j$ is optimal value of $\hat{\phi}_j; j = 1, 2, \ldots, p$ we have

\[ \tilde{\phi}_j \geq \phi^*_j \quad \text{or} \quad \tilde{\phi}_j - \phi^*_j \geq 0; j = 1, 2, \ldots, p \] \[ \text{(4.2)} \]

The increase in the value of the objective function $\hat{\phi}_j$, that is, $\tilde{\phi}_j - \phi^*_j$ is due to the use of a compromise allocation instead its individual optimum allocation. Let $x_j$ denote the tolerance limit for the increase $(\tilde{\phi}_j - \phi^*_j)$

\[ \Rightarrow \quad \tilde{\phi}_j - \phi^*_j \leq x_j; j = 1, 2, \ldots, p. \]

The GPP may now be stated as:

"Find $n_{c} = (n^*_{1c}, n^*_{2c}, \ldots, n^*_{Lc})$ such that the increase in $\phi^*_j$ for not using its optimum allocation, is less than or equal to $x_j$ for all $j = 1, 2, \ldots, p". The tolerance limits $x_j$ are the goal variables.

A reasonable compromise criterion will then be to choose $n_{hj}(c); h = 1, 2, \ldots, L$ that minimizes the total increase $\sum_{j=1}^{p} x_j$ for not using the individual optimum allocations. The required compromise allocation $n^*_c = (n^*_{1c}, n^*_{2c}, \ldots, n^*_{Lc})$ will then be the solution to the following GPP.

Minimize $\sum_{j=1}^{p} x_j$

subject to $\alpha_1 \left[ \sum_{h=1}^{L} W_{hj}^2 s_{hj}^2 \left( \frac{n_{hj}}{n_{hj}^2 - 1} \right) \right] + \alpha_2 \left[ \left( \sum_{h=1}^{L} \frac{W_{hj}^4}{n_{hj}} \frac{n_{hj}^2}{(n_{hj}^2 - 1)^2} \right) \left( C_{hj}^4 - \frac{(s_{hj}^2)^2}{(n_{hj}^2 - 1)^2} \right) \right]^{1/2} - x_j \leq \phi^*_j; j = 1, 2, \ldots, p
The GPP (4.3) may be solved by using the optimization software LINGO.

5 Numerical Illustration
To illustrate the proposed method numerically artificial data are used. A stratified population of size 1060 with four strata and two characteristics is considered. It is assumed that the total available budget \( A = 3500 \) units with an overhead cost \( c_0 = 500 \) units. Assume that the estimation of the two population means is of interest. The available amount for measurements and travel \( C_0 = A - c_0 = 3500 - 500 = 3000 \). This amount is bifurcated in proportion to

\[
\frac{\sum_{h=1}^{4} \bar{c}_h}{\sum_{h=1}^{4} \bar{c}_h} \approx \frac{C_01}{C_02} \text{ for the first and second characteristics respectively.}
\]

Table 1 gives values of strata sizes, strata weights, sample variances and the fourth moments about means for the two characteristics while Table 2 gives the expected measurement and travel costs and their estimated variances.

**Table 1**
Data with four strata and two characteristics

**Table 2**
Expected cost with estimates of their variances

For the sake of simplicity it is assume that the weights \( \alpha_1 = \alpha_2 = 0.5 \) and \( \beta_1 = \beta_2 = 0.5 \). The AINLPP (4.1) for \( j = 1 \) with data values from Table 1 and Table 2 will be

Minimize \( \hat{\phi}_1 = 0.5 \left\{ \frac{0.0094562122}{(n_{11} - 1)} + \frac{0.396891242}{(n_{21} - 1)} + \frac{0.0069775721}{(n_{31} - 1)} + \frac{0.309184763}{(n_{41} - 1)} \right\} 

- \left\{ 3.78248E - 05 \left( \frac{n_{11}}{n_{11} - 1} \right) + 1.7256141E - 03 \left( \frac{n_{21}}{n_{21} - 1} \right) 

+ 2.49199E - 05 \left( \frac{n_{31}}{n_{31} - 1} \right) + 1.0306159E - 03 \left( \frac{n_{41}}{n_{41} - 1} \right) \right\} \right\} \]
\[+0.5 \left\{ \left( \frac{1.890505E - 04}{n_{11}(n_{11} - 1)^2} + \frac{0.574865082}{n_{21}(n_{21} - 1)^2} + \frac{9.73730E - 05}{n_{31}(n_{31} - 1)^2} + \frac{0.123188972}{n_{41}(n_{41} - 1)^2} \right) \right. \]
\[\left. - \left( 3.0E - 09 \left( \frac{n_{11}}{(n_{11} - 1)^{2}} \right) + 1.08670E - 05 \left( \frac{n_{21}}{(n_{21} - 1)^{2}} \right) \right) \right. \]
\[\left. + 1.2E - 09 \left( \frac{n_{31}}{(n_{31} - 1)^{2}} \right) + 1.3688E - 06 \left( \frac{n_{41}}{(n_{41} - 1)^{2}} \right) \right) \right\}^{1/2} \]
subject to \[0.5 \left( 25n_{11} + 30n_{21} + 15n_{31} + 25n_{41} \right) + \left( 3\sqrt{n_{11}} + 2\sqrt{n_{21}} + 4\sqrt{n_{31}} + 5\sqrt{n_{41}} \right) \]
\[+ 0.5 \sqrt{\left( 14n_{12}^{2} + 16n_{22}^{2} + 13n_{32}^{2} + 15n_{42}^{2} \right) + \left( 2n_{11} + 3n_{21} + 2n_{31} + 4n_{41} \right) \leq 1440} \]
\[2 \leq n_{11} \leq 250 \]
\[2 \leq n_{21} \leq 230 \]
\[2 \leq n_{31} \leq 280 \]
\[2 \leq n_{41} \leq 300 \]
and \[n_{h1} \text{ integers; } h = 1, 2, 3 \text{ and } 4 \]

The optimum solution to the AINLPP (5.1) using optimization software LINGO is found to be

\[n_{11}^* = 8, n_{21}^* = 44, n_{31}^* = 9, n_{41}^* = 37 \text{ with } \phi_1^* = 0.01015286.\]

For \(j = 2\) the AINLPP (4.1) takes the form:

Minimize \(\dot{\phi}_2 = 0.5 \left\{ \left( \frac{0.017799928}{(n_{12} - 1)} + \frac{0.29282648}{(n_{22} - 1)} + \frac{0.105361338}{(n_{32} - 1)} + \frac{0.436543253}{(n_{42} - 1)} \right) \right. \]
\[\left. - \left\{ 7.11997E - 05 \left( \frac{n_{12}}{n_{12} - 1} \right) + 1.273289E - 03 \left( \frac{n_{22}}{n_{22} - 1} \right) \right\} \right. \]
\[\left. + 3.762905E - 04 \left( \frac{n_{32}}{n_{32} - 1} \right) + 1.4551442E - 03 \left( \frac{n_{42}}{n_{42} - 1} \right) \right) \right\} \]
\[+ 0.5 \left\{ \left( \frac{6.11397E - 04}{n_{12}(n_{12} - 1)^2} + \frac{0.29303844}{n_{22}(n_{22} - 1)^2} + \frac{0.02551245}{n_{32}(n_{32} - 1)^2} + \frac{0.219281427}{n_{42}(n_{42} - 1)^2} \right) \right. \]
\[\left. - \left( 9.8E - 09 \left( \frac{n_{12}}{(n_{12} - 1)^{2}} \right) + 5.5395E - 06 \left( \frac{n_{22}}{(n_{22} - 1)^{2}} \right) \right) \right\} \]
The optimum solution to the AINLPP (5.2) is:

\[ n_{12}^* = 9, n_{22}^* = 35, n_{32}^* = 23, n_{42}^* = 44 \text{ with } \phi_2^* = 0.01305923. \]

After having the optimum values of \( \phi_j^*; j = 1 \text{ and } 2 \), the GPP (4.3) for obtaining the required compromise allocation \( \bar{n}_i^* \) becomes

Minimize \( x_1 + x_2 \)

subject to 0.5 \[ \begin{bmatrix} 0.0094562122 \left(n_{1(c)-1}^{-1}\right) + 0.396891242 \left(n_{2(c)-1}^{-1}\right) + 0.0069775721 \left(n_{3(c)-1}^{-1}\right) + 0.309184763 \left(n_{4(c)-1}^{-1}\right) \end{bmatrix} \]

\[ - \left\{ 3.78248E-05 \left(\frac{n_{1(c)}}{n_{1(c)} - 1}\right) + 1.7256141E - 03 \left(\frac{n_{2(c)}}{n_{2(c)} - 1}\right) \right\} \]

\[ + 2.49199E - 05 \left(\frac{n_{3(c)}}{n_{3(c)} - 1}\right) + 1.0306159E - 03 \left(\frac{n_{4(c)}}{n_{4(c)} - 1}\right) \]  

\[ + 0.5 \left\{ \left(\frac{1.890505E - 04}{n_{1(c)}(n_{1(c)} - 1)} + \frac{0.574865082}{n_{2(c)}(n_{2(c)} - 1)} + \frac{9.73730E - 05}{n_{3(c)}(n_{3(c)} - 1)} + \frac{0.123188972}{n_{4(c)}(n_{4(c)} - 1)} \right) \right\} \]

\[ - \left( 3.0E - 09 \left(\frac{n_{1(c)}}{(n_{1(c)} - 1)^2}\right) + 1.08670E - 05 \left(\frac{n_{2(c)}}{(n_{2(c)} - 1)^2}\right) \right) \]

\[ + 1.2E - 09 \left(\frac{n_{3(c)}}{(n_{3(c)} - 1)^2}\right) + 1.3688E - 06 \left(\frac{n_{4(c)}}{(n_{4(c)} - 1)^2}\right) \] \[ - x_1 \leq 0.01015286 \]

0.5 \[ \begin{bmatrix} 0.017799928 \left(\frac{1}{n_{1(c)} - 1}\right) + 0.292842648 \left(\frac{1}{n_{2(c)} - 1}\right) + 0.105361338 \left(\frac{1}{n_{3(c)} - 1}\right) + 0.436543253 \left(\frac{1}{n_{4(c)} - 1}\right) \end{bmatrix} \]
\[
- \left\{ 7.11997E - 05 \left( \frac{n_{1(c)}}{n_{1(c)} - 1} \right) + 1.2732289E - 03 \left( \frac{n_{2(c)}}{n_{2(c)} - 1} \right) \\
+ 3.762905E - 04 \left( \frac{n_{3(c)}}{n_{3(c)} - 1} \right) + 1.4551442E - 03 \left( \frac{n_{4(c)}}{n_{4(c)} - 1} \right) \right\}
+ 0.5 \left\{ \left( 6.113973E - 04 \frac{n_{1(c)}}{n_{1(c)} - 1} + 0.29303844 \frac{n_{2(c)}}{n_{2(c)} - 1} \right) + 0.025511245 \frac{n_{3(c)}}{n_{3(c)} - 1} + 0.219281427 \frac{n_{4(c)}}{n_{4(c)} - 1} \right\}
- 9.8E - 09 \left( \frac{n_{1(c)}}{n_{1(c)} - 1} \right) + 5.5395E - 06 \left( \frac{n_{2(c)}}{n_{2(c)} - 1} \right)
+ 3.254E - 07 \left( \frac{n_{3(c)}}{n_{3(c)} - 1} \right) + 2.4365E - 06 \left( \frac{n_{4(c)}}{n_{4(c)} - 1} \right) \right\}^{1/2}
- x_2 \leq 0.01305923
\]

\[
0.5 \left[ (55n_{1(c)} + 62n_{2(c)} + 34n_{3(c)} + 47n_{4(c)}) + (3\sqrt{n_{1(c)}} + 2\sqrt{n_{2(c)}} + 4\sqrt{n_{3(c)}} + 5\sqrt{n_{4(c)}}) \right]
+ 0.5 \sqrt{(31n_{1(c)}^2 + 31n_{2(c)}^2 + 31n_{3(c)}^2 + 32n_{4(c)}^2) + (2n_{1(c)} + 3n_{2(c)} + 2n_{3(c)} + 2n_{4(c)}) \leq 3000}
\]

\[
2 \leq n_{1(c)} \leq 250
\]

\[
2 \leq n_{2(c)} \leq 230
\]

\[
2 \leq n_{3(c)} \leq 280
\]

\[
2 \leq n_{4(c)} \leq 300
\]

\[
h_{(c)} \text{ integers; } h = 1, 2, 3 \text{ and } 4
\]

and \( x_1 \geq 0, x_2 \geq 0 \)

Using LINGO, the optimum values of are given as:
\[
n_{1(c)}^* = 8, n_{2(c)}^* = 39, n_{3(c)}^* = 22, n_{4(c)}^* = 42
\]
\[
x_1^* = 0.00000 \text{ and } x_2^* = 0.0001222513
\]

with objective value 0.00001222635.
The estimated variances of the estimators \( \hat{y}_{1st} \) and \( \hat{y}_{2st} \), that is \( \hat{V}_c(\hat{y}_{1st}) \) and \( \hat{V}_c(\hat{y}_{2st}) \) worked out under the proposed compromise allocation are
\[
\hat{V}_c(\hat{y}_{1st}) = 0.019037429, \hat{V}_c(\hat{y}_{2st}) = 0.024916815 \text{ respectively}
\]
and \( \text{Trace} = 0.019037429 + 0.024916815 = 0.043954244 \).
The total sample size is given as
\[
\sum_{h=1}^{4} n_h = 111.
\]

6 Comparison with some other allocations
In this section the performance of the proposed compromise allocation has been compared with that of Proportional and Cochran’s Average allocations
assuming the same data as deterministic.

6.1 Proportional allocation
The proportional allocation is given by

\[ n_h = nW_h; \quad h = 1, 2, ..., L \]  \hspace{1cm} (5.4)

where \( n = \sum_{h=1}^{L} n_h \) is the total sample size.

Taking \( n = \sum_{h=1}^{L} n_h^{(c)} = 111 \), the rounded off proportional allocation is obtained as:

\[ n_{1\text{prop}} = 26, \quad n_{2\text{prop}} = 24, \quad n_{3\text{prop}} = 29, \quad n_{4\text{prop}} = 32 \]

with \( \hat{V}_{\text{prop}}(\bar{y}_{1st}) = 0.026803465 \) and \( \hat{V}_{\text{prop}}(\bar{y}_{2st}) = 0.030161516 \),

and \( \text{Trace} = 0.026803465 + 0.030161516 = 0.056964981 \).

6.2 Cochran’s Average allocation
In the sampling literature, for compromise allocation in multivariate stratified surveys the Cochran’s Average Allocation is well known (see Cochran (1977)). For the present problem this compromise allocation may be obtained by using the following formula

\[ n_{h(a)} = \frac{1}{p} \sum_{j=1}^{p} n_{hj}^*; \quad h = 1, 2, ..., L \]  \hspace{1cm} (5.5)

where the suffix ‘a’ stands for ‘Average Allocation’ and \( n_{hj}^*; \quad h = 1, 2, ..., L; \quad j = 1, 2, ..., p \) are the individual optimum allocations obtained by solving the \( p \)-deterministic equivalents of SINLPP given in (3.15). Using (5.5) the rounded off Cochran’s Average Allocation for the data provided in Table1 and Table 2 may be obtained by averaging over \( j = 1, 2 \) the solution obtained to AINLPP (5.1) and (5.2) are as follows:

\[ \mathbf{n}_1^* = (n_{11}^*, n_{21}^*, n_{31}^*, n_{41}^*) \]

\[ = (8, 44, 9, 37) \]

and \( \mathbf{n}_2^* = (n_{12}^*, n_{22}^*, n_{32}^*, n_{42}^*) \)

\[ = (9, 35, 23, 44) \]

This gives
\[ \underline{n}_{(a)} = (n_{1(a)}, n_{2(a)}, n_{3(a)}, n_{4(a)}) \]

where \( \underline{n}_{h(a)} = \frac{n_{h1} + n_{h2}}{2} ; h = 1, 2, 3 \& 4. \)

So that after rounding off \( \underline{n}_{(a)} = (8, 39, 16, 40) \)

with estimated variances are \( \hat{V}_{1(a)} = 0.019524442, \hat{V}_{2(a)} = 0.027232442. \) Where the suffix ‘a’ stands for ‘Average Allocation’.

The ‘Trace’ is equal to \( 0.019524442 + 0.027232442 = 0.046756884. \)

If \( Trace(\underline{n}) \) and \( Trace(\underline{n}’) \) represent the traces of the variance-covariance matrices of the stratified sample mean \( \bar{y}_{jst} \) under two different allocations \( \underline{n} = (n_1, n_2, ..., n_L) \) and \( \underline{n}’ = (n_1’, n_2’, ..., n_L’) \) then the relative efficiency \( (R.E.) \) of the allocation \( \underline{n} \) with respect to \( \underline{n}’ \) may be determined by

\[ R.E. = \frac{Trace(\underline{n}’)}{Trace(\underline{n})} \]  

(see Sukhatme (1984)).

7 Conclusions

Table 3 summarizes the results obtained for the three allocations, ‘Proportional’, ‘Cochran’ and ‘Proposed’.

<table>
<thead>
<tr>
<th>Table 3</th>
<th>Summary of results</th>
</tr>
</thead>
</table>

The last column of the Table 3 gives the relative efficiencies of the two compromise allocations with respect to the proportional allocation. An observation of the relative efficiencies reveals that the proposed allocation is the best among the considered allocations.

Acknowledgement: The authors are grateful to the Editor-in-Chief and the learned Reviewers for their comments and suggestions which helped in improving the manuscript in its present form.

References


LINGO. (2001). LINGO-Users Guide. Published by LINDO SYS-
TEM INC., 1415, North Dayton Street, Chicago, Illinois, 60622, USA.


Table 1
Data with four strata and two characteristics

<table>
<thead>
<tr>
<th>$h$</th>
<th>$N_h$</th>
<th>$W_h$</th>
<th>$s^2_{h1}$</th>
<th>$s^2_{h2}$</th>
<th>$C^4_{h1}$</th>
<th>$C^4_{h2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>250</td>
<td>0.24</td>
<td>0.17</td>
<td>0.32</td>
<td>0.09</td>
<td>0.30</td>
</tr>
<tr>
<td>2</td>
<td>230</td>
<td>0.22</td>
<td>8.43</td>
<td>6.22</td>
<td>330.41</td>
<td>170.89</td>
</tr>
<tr>
<td>3</td>
<td>280</td>
<td>0.26</td>
<td>0.10</td>
<td>1.51</td>
<td>0.03</td>
<td>7.52</td>
</tr>
<tr>
<td>4</td>
<td>300</td>
<td>0.28</td>
<td>3.86</td>
<td>5.45</td>
<td>34.10</td>
<td>63.88</td>
</tr>
</tbody>
</table>

Table 2
Expected cost with estimates of their variances

<table>
<thead>
<tr>
<th>$h$</th>
<th>$c_{h1}$</th>
<th>$c_{h2}$</th>
<th>$c_h = c_{h1} + c_{h2}$</th>
<th>$\sigma^2_{c_{h1}}$</th>
<th>$\sigma^2_{c_{h2}}$</th>
<th>$\sigma^2_{c_h} = \sigma^2_{c_{h1}} + \sigma^2_{c_{h2}}$</th>
<th>$t_h$</th>
<th>$\sigma^2_{t_h}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>25</td>
<td>30</td>
<td>55</td>
<td>14</td>
<td>17</td>
<td>31</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>30</td>
<td>32</td>
<td>62</td>
<td>16</td>
<td>15</td>
<td>31</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>15</td>
<td>19</td>
<td>34</td>
<td>13</td>
<td>18</td>
<td>31</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>25</td>
<td>22</td>
<td>47</td>
<td>15</td>
<td>17</td>
<td>32</td>
<td>5</td>
<td>4</td>
</tr>
</tbody>
</table>

Table 3
Summary of results

<table>
<thead>
<tr>
<th>Allocations</th>
<th>$n_1$</th>
<th>$n_2$</th>
<th>$n_3$</th>
<th>$n_4$</th>
<th>Trace</th>
<th>R.E. w.r.t proportional allocational</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proportional</td>
<td>26</td>
<td>24</td>
<td>29</td>
<td>32</td>
<td>0.056964981</td>
<td>1.0000</td>
</tr>
<tr>
<td>ochran</td>
<td>8</td>
<td>39</td>
<td>16</td>
<td>40</td>
<td>0.046756884</td>
<td>1.218322868</td>
</tr>
<tr>
<td>Proposed</td>
<td>8</td>
<td>39</td>
<td>22</td>
<td>42</td>
<td>0.043954244</td>
<td>1.296006388</td>
</tr>
</tbody>
</table>
CONTENTS

On the number of solutions of the Diophantine equation
\[ y^2 = nx (Ax^2 \pm C); \]
\textit{Wenquan Wu, Alain Togbe, Bo He & Shichun Yang} 1-16

On a family of permutation polynomials of \( F_q; \)
\textit{Kacem Belghaba & Salima Kebli} 17-29

Expectation Identities from extended Burr XII Distribution based on generalized order Statistics and Characterization;
\textit{Devendra Kumar & Anju Goyal} 30-46

A Bootstrap Approach to Error-Reduction of Nonlinear Regression Parameters Estimation;
\textit{H. O. Obiora-Ilouno & J. I. Mbegbu} 47-58

New Formula for Fractional Differentiation;
\textit{Kuldeep Singh Gehlot & Jyotindra C. Prajapati} 59-66

Compromise Allocation in Multivariate Stratified Sampling: A Stochastic Programming Approach;
\textit{Ummatul Fatima, Rahul Varshney & M. J. Ahsan} 67-86